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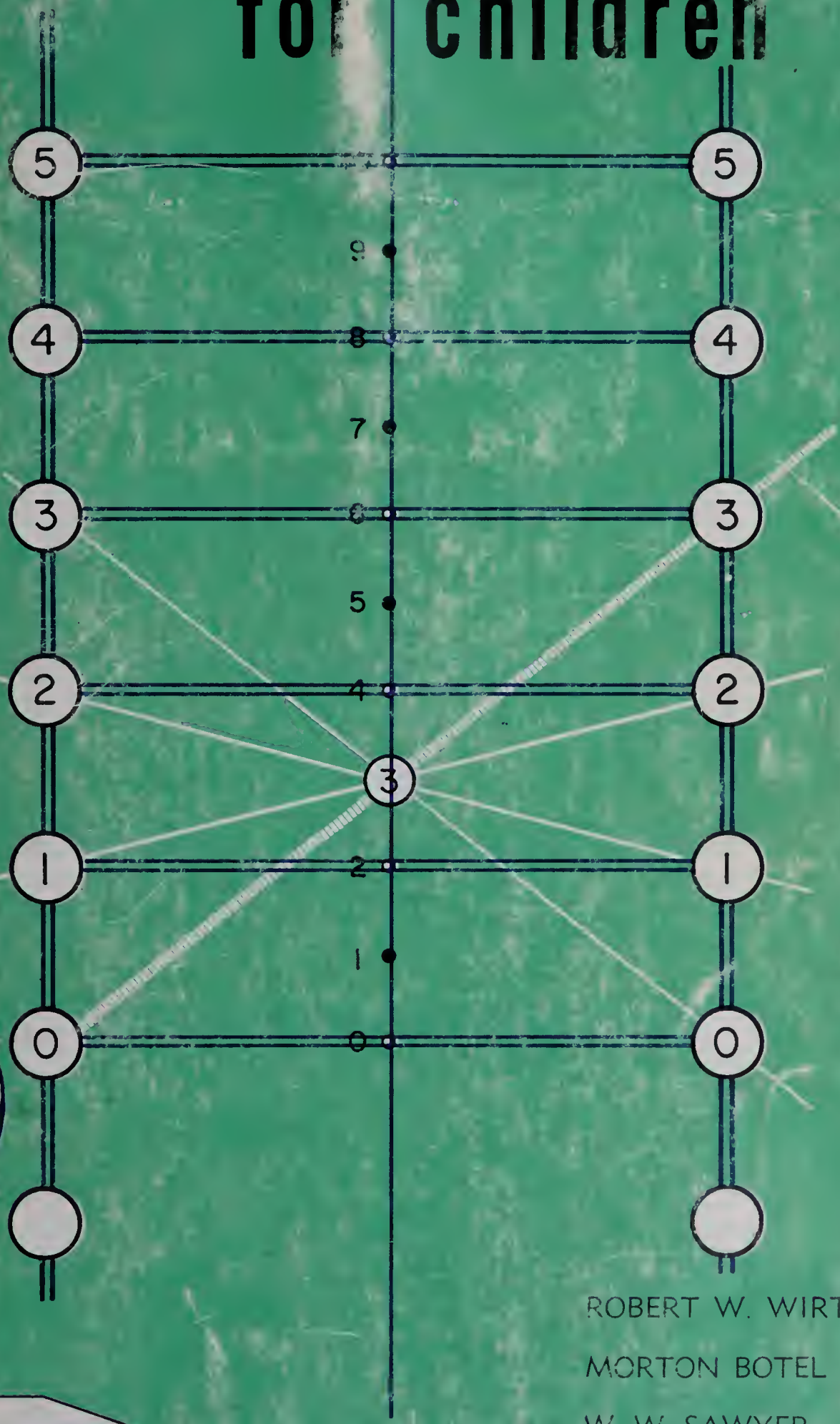
math workshop for children



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ROBERT W. WIRTZ
MORTON BOTEL
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Robert W. Wirtz

Assistant Professor of Education, University of Illinois
Staff Research Associate, Addressograph-Multigraph Corporation
Mathematics Consultant, Bucks County Schools, Pennsylvania

Morton Botel

Assistant Superintendent and Reading Consultant, Bucks County Schools, Pennsylvania

W. W. Sawyer

Professor of Mathematics, Wesleyan University, Connecticut

Marvin Levitt

Numo drawings and Art Direction

Art Staff:

Lieselotte Esser, Dorothy Kurr, Roy Lipschutz

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THE SCIENCE FAIR PROJECT OF ALEC MARSON

It was truly an odd looking machine that Alec Marson entered in the Fairview School Science Fair. When the principal, Mr. Wilson, looked for Alec, he was gone. He had left town with his family.

"No one knows where they have gone," Mr. Wilson explained to the judges of the Fair. "In fact, no one knows where they came from."

The judges were puzzled by Alec's machine.

"What does it do?" one asked.

"What is it?"

"Maybe we could open it up to see what's inside."

But the machine had no covers — no screws to take out. It seemed as if it were made out of a single piece of strange metal.

An Experiment

"Well, let's see if anything happens when we write on the green screen that's marked 'You write here'," said one of the judges. "I suppose I could use that thing that looks like a pencil and is wired in the machine."

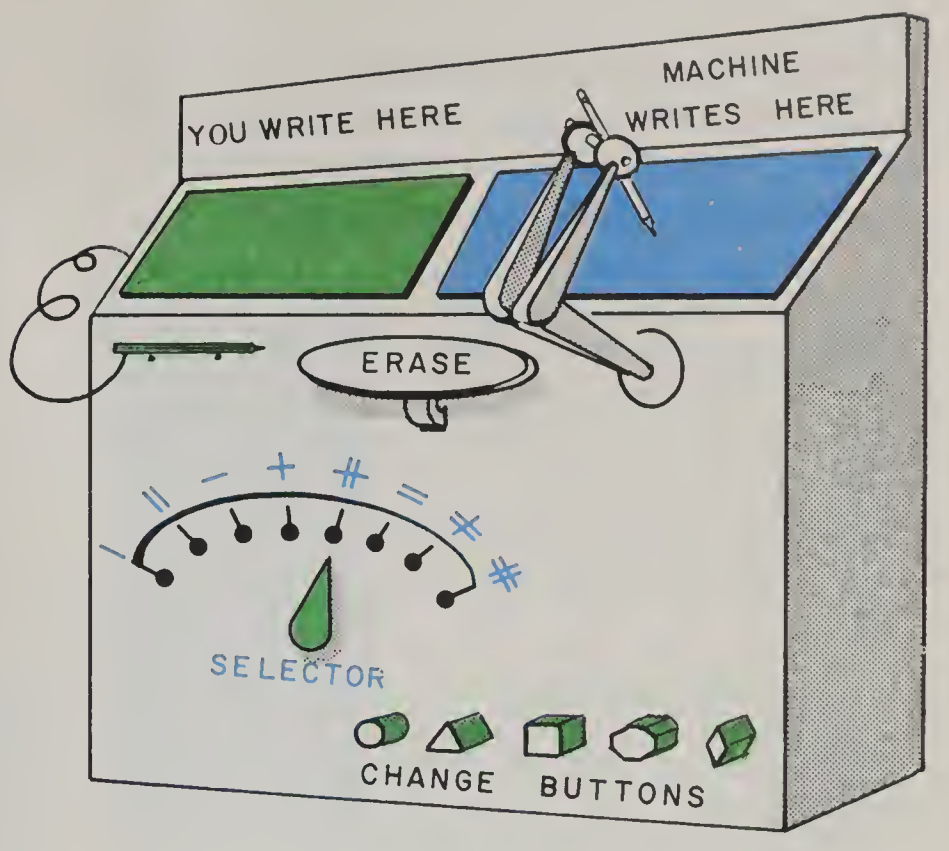
He wrote a large "2" on the screen. Without a sound, the two arms at the right of the machine moved the other pencil around on the blue screen and wrote "4."

"Well, it moves. Something happened; but I'm not sure I know any more than before."

"Let me try it," another judge added. He pushed the "ERASE" key, and the green and blue screens went blank. Then he wrote "0" on the green screen — and the machine wrote "0" on the blue screen.

Another judge pushed the "ERASE" key, and then wrote "5" on the green screen.

At once, the machine wrote "10" on the blue screen.



"That's interesting," Mr. Wilson said. "Let's keep a record of these experiments."

He made a chart on the board to keep track of the results. Here are part of the results. Can you guess the rest?

You write	2	0	5	1	7	25	
Machine writes	4	0	10	2			18

Judge Harris said, "I think the machine adds 5 to any number written; then doubles the result; and, finally, subtracts 10. I see it this way."

Machine does:

You write	add 5	double	subtract 10	Machine writes
2	7	14	4	4
0	5			0
5		20		10
1				2
7				
25				
				18
23				
	16			

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ANOTHER SUGGESTION

Judge Thornton said, "Perhaps the machine multiplies the number you have written by 6, and then divides the result by 3. If it did, the results would be:"

You write	Machine's Work		Machine writes
	multiply by 6	divide by 3	
2			
0			
5			
1			
7			
25			

Mr. Wilson, the principal, suggested that perhaps the machine wasn't as complicated as the judges thought.

"Maybe it simply reads what you write and then doubles that number."

But Judge Jones was ready with another explanation.

"It could be that the machine adds 3, triples that sum, subtracts 8 from the result, and then takes away 1 more than the number you write."

You write	add 3	triple	subtract 8	subtract 1 more than no.	Mach. writes
2	5	15	7	4	
0	3	9			
5	8				
1					
7					
25					
9					
	26				
		42			
			31		
		60		34	
	53				100

MORE EXPLANATIONS

Other suggestions were as follows:

a. "Multiply by 10 and divide by 5."

b. "Multiply by 3 and subtract the number you wrote."

c. "Halve the number you wrote and multiply the result by 4."

d. "Subtract 5; double the result, and then add 10." (Do you think the machine can subtract 5 from 0 or from 1 or 2 or 3 or 4?)

Mr. Wilson was a little impatient. "I'm sure," he said, "that Alec Marson could have built a machine that got the same results by an even more complicated procedure.

"He might have had his machine add 2 to the number you write; multiply by 6, add 9, divide by 3 and subtract 7. Here are some results:"

You write	2	0	5	1	7	3	4
add 2	4						
multiply by 6	24						
add 9	33						
divide by 3	11						
subtract 7	4						
Machine writes	4						

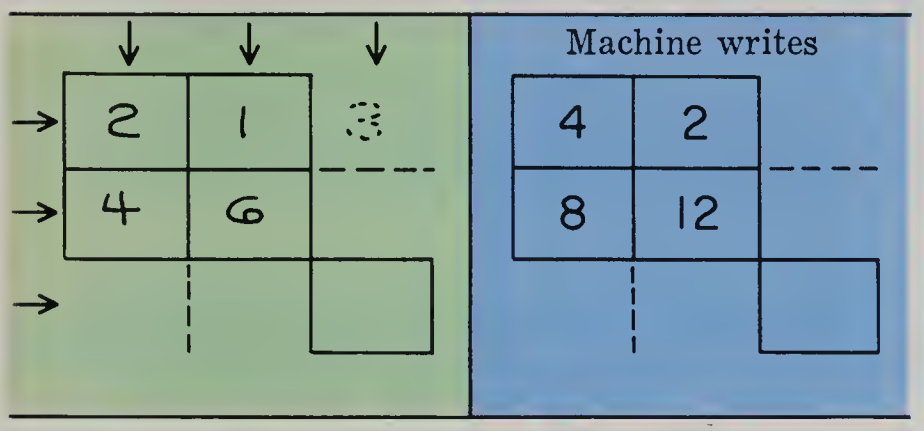
"However," Mr. Wilson continued, "Alec Marson was a very bright boy. He seldom used complicated means when the same thing could be done more easily.

"I'm convinced he built a machine that doubles any number you write on the green screen.

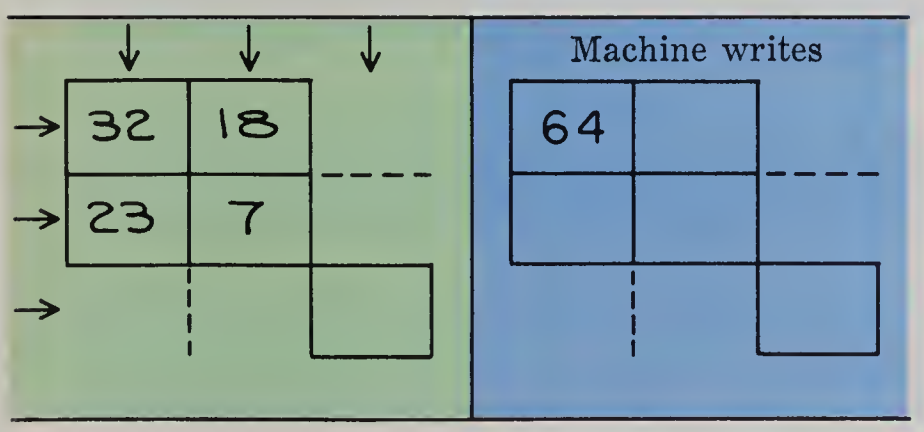
"I think that there is a lot more to Alec's machine than we know about yet. Let's see what else it will do."

Mr. Wilson explained that Alec had thought Cross Number Puzzles were fun. The judges had never heard of such puzzles.

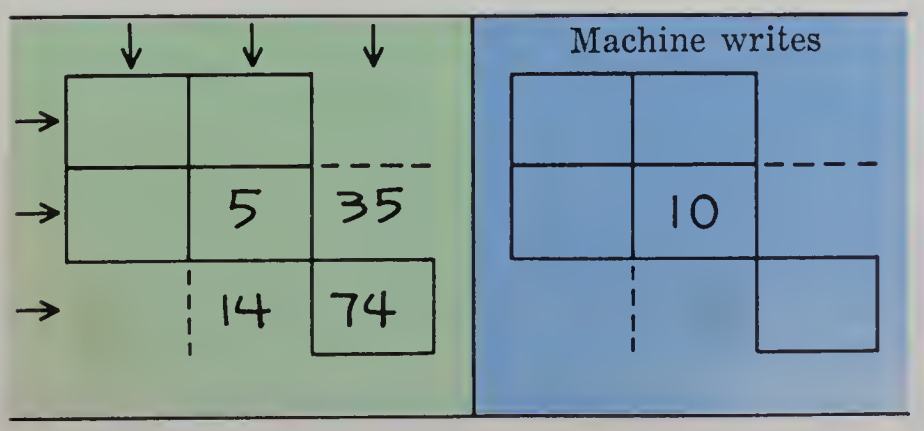
"In one form," Mr. Wilson explained, "four numbers are written in a group of four boxes. You add each row and each column separately, and then add each pair of sums." Mr. Wilson illustrated by drawing on the green screen . . . and, to everyone's amazement, the machine started writing on the blue screen.



"Amazing!" the judges said in unison. Judge Thornton wanted to try a cross number puzzle with larger numbers.



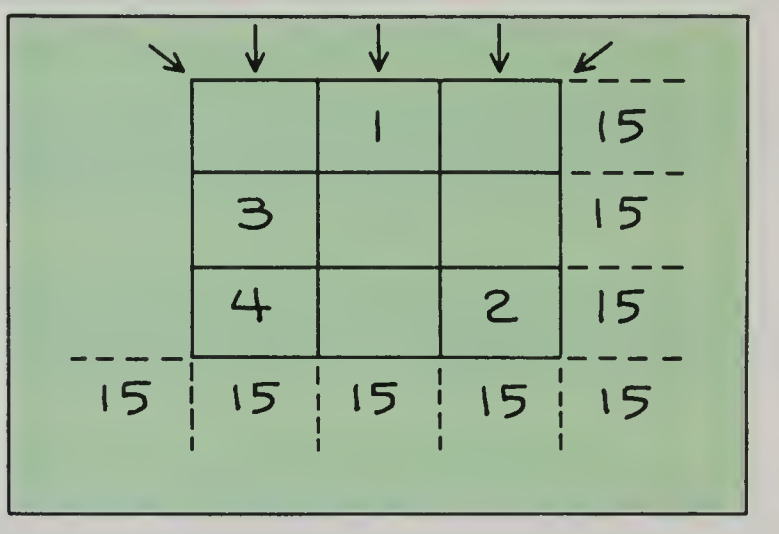
Mr. Wilson said, "Alec knew how to complete puzzles when the numbers were written differently. Perhaps the machine knows how, too."



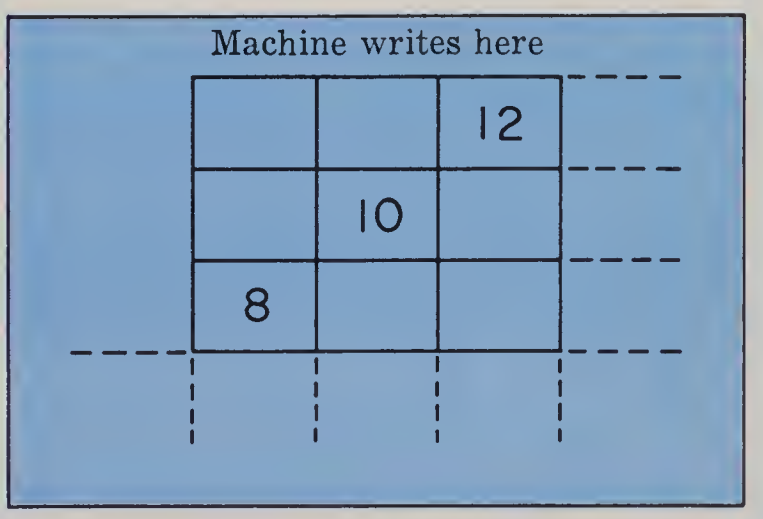
Judge Jones shook his head. "That boy certainly was unusual."

"Yes," replied Mr. Wilson, "he was, indeed. He also enjoyed working with Magic Squares in which the sum of every row, column and major diagonal is the same number. I wonder what would happen if I put a Magic Square on the green screen."

Mr. Wilson started a Magic Square — and the machine completed it for him. Can you?



And, at the same time, the machine was writing on the blue screen. Can you guess how it finished the Magic Square below:



Judge Thornton said, "I agree with Mr. Wilson . . . the machine finds a way to double everything. Let's call it a Doubling Machine."

"Not yet," Mr. Wilson suggested. "I want to know what happens when we move the pointer along the dial marked Selector."

Mr. Wilson turned the pointer from # to #. Then he wrote 4 on the green screen — and the machine wrote 16 on the blue screen.

"Ah!" exclaimed Judge Thornton, "now it's multiplying by 4. This is truly a wonderful machine."

After pushing the Erase key, Mr. Wilson wrote 6 and the machine wrote 18.

"What's that? Did the machine make a mistake?"

Mr. Wilson made a record of his results. Can you complete the blanks?

You write	4	6	12	9	13	24	45
Machine writes	16	18	24	21	25		

"No," Mr. Wilson concluded, "the machine did not make a mistake."

"I think I see what's happening," Mr. Thornton said. "The machine adds 6 to the number written, doubles the result and subtracts the number you wrote. I would show it this way:"

You write	4	6	12	9	13	24	45
add 6	10	12					
double	20	24					
subtract no. you wrote	16						
Machine writes	16	18	24				

Judge Jones had been scratching his head. He was puzzled by their results. And then his face lit up.

"I've got it!" he said. "The machine doubles the number written, adds 24 to the result, and then divides by 2. Like this:"

You write	4	6	12	9	13	24	25
double	8						
add 24	32						
divide by 2							
Machine writes							

By this time, Judge Harris had his own idea of the way the machine worked.

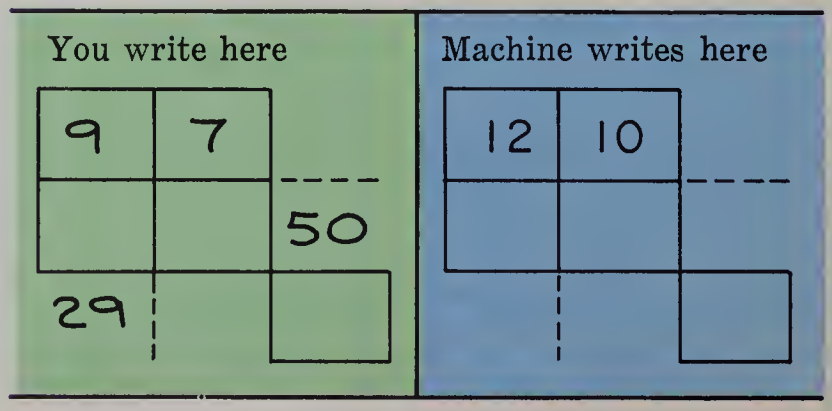
"Add 4," he said. "Triple the result; subtract twice the number you wrote. Like this:"

You write	4	6	12	9	13	24	25
add 4	8	10					
triple	24	30					
subtract twice no. you wrote		30 -12					
Machine writes		18					

Mr. Wilson was growing impatient again.

"No," he said. "Alec always did things the easy way. He avoided complications. I'm sure that when the selector points to #, the machine reads the number you wrote, adds 12 to it and writes the result."

"Yes, of course," Judge Harris admitted. "We certainly missed the easy way. Let's try another Cross Number Puzzle."



Experimenting with the Selector.

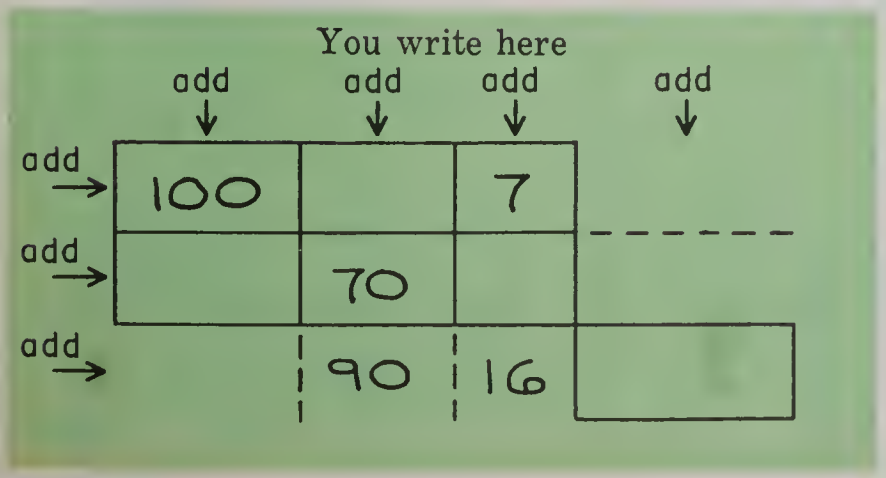
The judges next moved the selector to each of the eight positions indicated.

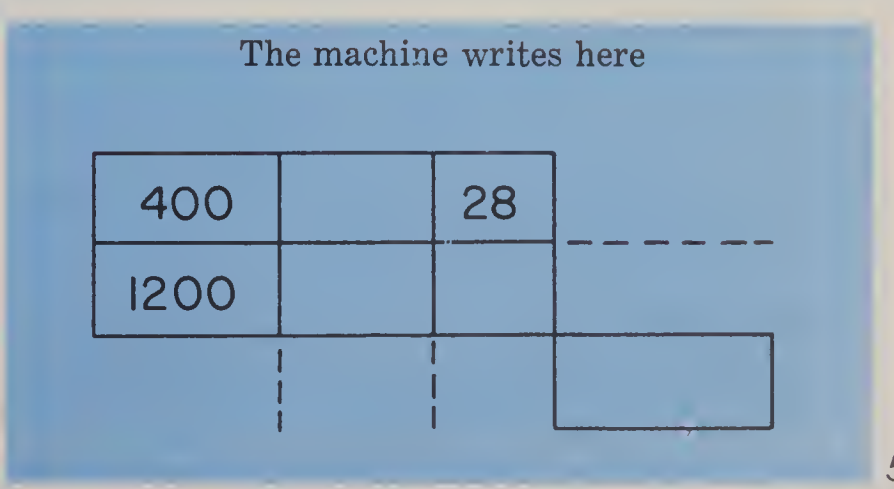
Mr. Wilson kept track of the results, some of which are shown in the chart below. Can you fill in the blanks? Work as far to the right as you can.

Selector position ↓

		a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
|	You write	1	2	5	14	10	9	0	8	50	75	106	500	610
	Machine writes	3	5	11	29									
||	You write	0	1	2	3	5	7	50	6	44	72	110	314	350
	Machine writes	3	6	9	12	18								
—	You write	10	2	7	1	11	3	8	5	15	25	31	200	320
	Machine writes	19	3	13	1	21								
+	You write	3	1	5	2		7		9		20	25	307	515
	Machine writes	18	6	30	12	24		36		48				
++	You write	0	7	1	4	19	36	49	28					199
	Machine writes	0	14	2	8					200	250	702	322	
=	You write	2	1	5	10	4				8	100	111	220	213
	Machine writes	9	5	21	41		41	37	13					
≠	You write	0	11	15	23	13		12						
	Machine writes	0	44	60	92		84		56	100	72	92	368	496
#	You write	1	7	8	48						108	318	490	899
	Machine writes	13	19	20	60	12	41	62	70	100				

When the pointer of Alec Marson's Science Fair project was at "≠," Mr. Wilson tried still another Cross Number Puzzle using some numbers larger than those used in previous examples. Only part of the results are given below. Can you fill in the blanks?



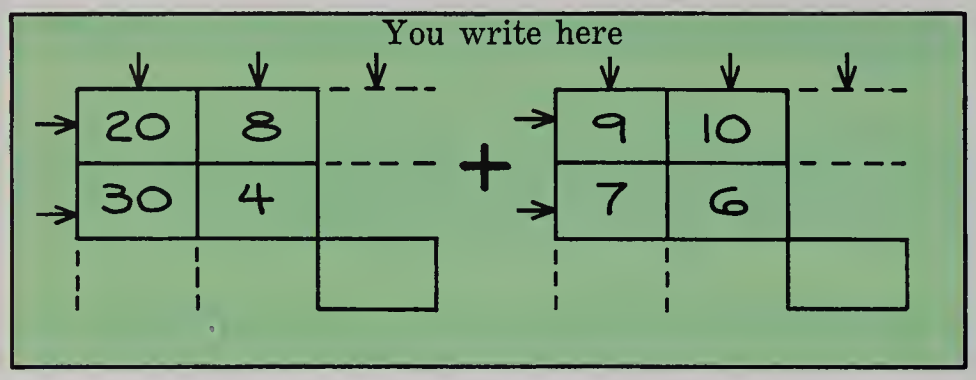


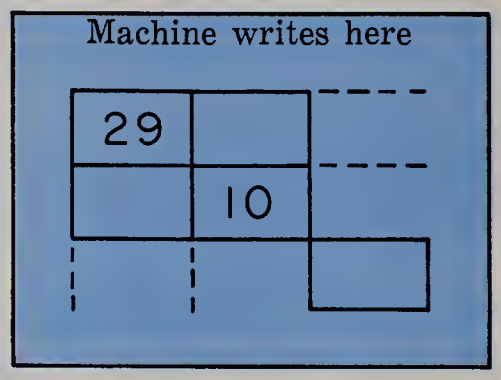
Mr. Wilson Does His Homework

The judges decided to adjourn until the next morning. Tired, but still curious, they left.

Mr. Wilson, the principal, was so curious, however, that when he left he took Alec's machine with him. He wanted to find out about those "Change Buttons."

In his study at home, he pushed the triangular button. Two Cross Number Puzzle forms flashed on the green screen and one on the blue screen. Mr. Wilson went to work.





Can you see a pattern that suggests a way to fill in the blanks?

Mr. Wilson continued his experiments. Below are shown some of the records he kept:

The green frames appeared on the green screen — the blue frames on the blue screen. Sometimes, Mr. Wilson used a vertical form.

A.

	13	
		36
71		100

+

	8	
		37
80		95

=

	21	
151		

B.

	7	
21		58

+

	25	
15		
23		59

=

25	18	

C.

		124
200		
	40	

+

300		351
	58	

=

	75	
300		
	149	

D.

45		
32	110	

+

29	125	
		58
57		

=

		549

After completing the five experiments on page 6, Mr. Wilson noticed that the Selector pointer was set at " | ." He noted this as one stroke down.

"I wonder what will happen if I move the pointer over to two-strokes-down, (||), he said to himself. He left the triangular button pressed down.

At once, two Cross Number Puzzle forms appeared on the green screen, with a minus sign between them.

M.

		15		2				9	12		
	15		−		3		=				
25		50		6		10		19			

N.

22					7	10			1		
		54	−	9			=	6			
	47									53	

O.

						8		4	18		
			−		8	12	=	26	32		
		100		10							

P.

	30	63									
20		37	−								
10	12										
11											
			=				=				=
23				100	25			60		79	
	4	13		300	27						
								350	63		

Q.

300		342									
			−								
800	91										
			=				=				=

R.

	80	405									
	106	581	−								
90	7										
	36										
			=				=				=
60		79		100	25						
				300	27						
350	63										

"Most interesting and original," Mr. Wilson thought. "I wish I could see inside Alec's machine."

Next, he moved the selector pointer to one-stroke-across — Magic Squares appeared where Cross Number Puzzle forms had been.

A

4	3		
		1	
2		6	
			15

+

B

		17	
19	11		
	15		
			33

=

A + B

	10		
28		4	48
		19	

C

	2		
6	10		
		4	
			30

+

D

	25		
			75

=

C + D

56		42	
	35	49	
28			

E

+

F

		16	
	20		
24	4		

=

E + F

14	63		
		21	
	7		
	105		

G

	7		
	9	13	
10			
	27		

+

H

	22		
		28	
25	26		
	72		

=

G + H

	29		
35		27	99

Mr. Wilson wondered to himself: "Can you add Magic Squares in much the same way you can add numbers?"

Mr. Wilson was very tired by now, but he was too fascinated by Alec's machine to stop. He moved the selector to one - stroke - across and one - stroke - down.

"Ah-hah!" he said aloud, "it's multiplication — and in the form of Cross Number Puzzles."

a.

2 ×				
	100	30	4	
			5	385
	100	40		
	500		15	

=

600	160		
		12	
1,000			

b.

5 ×				
	100		3	
			6	

=

500	450		995
	100		615
500			780
1,500	800		

c.

3 ×				
		40	5	
			8	
		120	15	735

=

600			
	60		
900			

d.

6 ×				
	200	20		
	300	60		
	200	60		
				856

=

		54	
1,200	360	30	
4,200	840		

e.

4 ×				
	1,000	70	8	
	1,500	110	21	

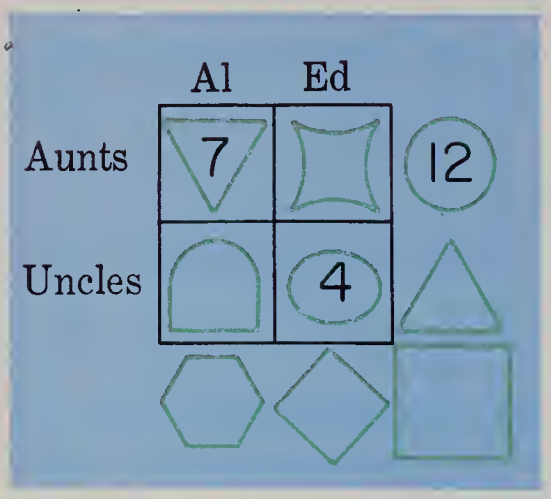
=

400		16	
	160		

Mr. Wilson's Dream

When finally he did go to bed, Mr. Wilson had a hard time getting to sleep. And when finally he did sleep, he had peculiar dreams.

On an immense blackboard, he saw a Cross Number Puzzle form. What could it mean?



Al and Ed were not related. Al had 7 aunts. Ed had 4 uncles. Together, they had 12 aunts. Altogether, Al had 10 aunts and uncles.

What statements are suggested by the following?

Ed has ____ aunts.

Al has ____ uncles.

Ed has _____

Together, Ed and Al have ___ uncles.

_____

_____

_____

_____

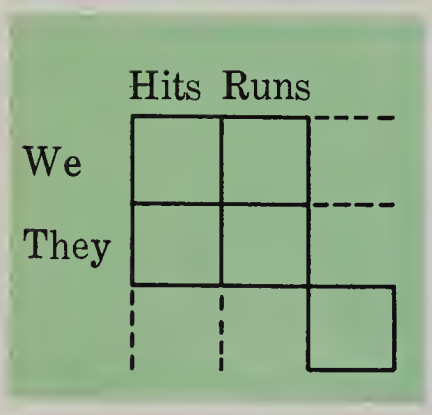
_____

_____

_____

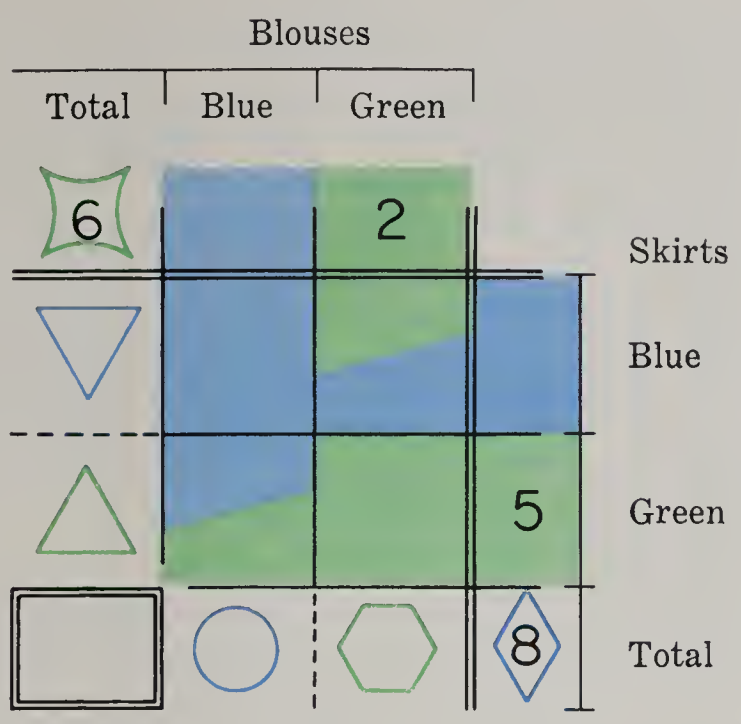
_____

Mr. Wilson's dream changed from aunts and uncles to baseball.



We won by a score of 11 to 9 runs. But they had 15 of the 28 hits that both teams made together. Fill in the Cross Number Puzzle to show what happened. Then make as many true statements as you can about the game.

Mr. Wilson dreamed of his wife selecting an outfit for a shopping trip.



She wanted to wear blue or green or both. She was counting her blouses and skirts. She counted out 2 green blouses and 5 green skirts. She also counted out a total of 6 blouses and 8 skirts.

"That's all the information I need," Mr. Wilson said in his dream. "I know how many combinations there are to choose from. The only thing I don't know is which of the outfits she will choose."

Can you complete the chart that Mr. Wilson completed?

Now you can complete the following statements about Mrs. Wilson's problem:

a. How many green blouses has she?____

b. How many blue blouses?____

c. How many blouses and skirts has she altogether from which to choose?____

d. How many different outfits can she wear that are all blue?____

e. How many outfits are green?____

f. How many with a green blouse?____

g. How many with a blue blouse?____

h. How many with a green skirt?____

i. How many with a blue skirt?____

j. How many outfits would be all green or all blue?____

k. How many not all green or all blue?____

l. How many outfits had she to select from all together?____

What ideas are suggested below?

A. △ With a green skirt, she can make _____ outfits.

B. ○ + ⬡ _____

C. □ _____

D. ⟡ _____

E. △ − ▽ _____

F. ○ − ⬡ _____

In greeting her arithmetic class the next morning, Mrs. Esser said, "Mr. Wilson has covered our blackboard with some Magic Squares. He asked me to find out whether or not the whole array makes any sense to you.

"Remember that in a Magic Square, the numbers in the cells are all different, but the sums of all rows, columns and diagonals are the same number."

Can you complete the first 5 examples so that they are "magic"?

Magic Numbers

A (15)

8		6
3	5	
4		2

B (36)

21		15
		18
	24	3

C (30)

8		
9	14	

D ( )

		30
15	25	35
20		10

E ( )

	2	
11	26	5

In what way can you say that A is a multiple of D?

A( ) + B( ) = Z( )

Z( )

29		21
	17	25
13		5

Z( ) − B( ) = Y( )

Y( )

		6
		7

Z( ) + Y( ) = X( )

X( )

		27
		32

C( ) + E( ) = W( )

W( )

D( ) − E( ) = V( )

V( )

B( ) + S( ) = U( )

U( )

B( ) + D( ) = T( )

T( )

W( ) − V( ) = S( )

S( )

U( ) − V( ) = R( )

R( )

"And what do you make of this?"

("A" is the first Magic Square on page 12)

MN is an abbreviation for Magic Number — the sum of each row, column, and diagonal.

2 x A

MN is____

16	2	
6		
8		4

5 x A

MN is____

7 x A

MN is____

9 x A

MN is____

6 x A

MN is____

3 x A

MN is____

8 x A

MN is____

4 x A

MN is____

12 x A

MN is____

21 x A

MN is____

11 x A

MN is____

10 x A

MN is____

Are the following statements true?

I- 2 x A + 5 x A = 7 x A _____

II- 9 x A − 6 x A = 3 x A _____

III- 8 x A + 4 x A = 12 x A _____

IV- 21 x A − 11 x A = 10 x A _____

V- 6 x A + 4 x A = 10 x A _____

Complete the following as true statements:

VI- 2 x A + 9 x A = _____

VII- 8 x A − 7 x A = _____

VIII- 5 x A + 5 x A = _____

IX- 2 x A + 6 x A + 4 x A = _____

X- 9 x A + 2 x A − 7 x A = _____

Mr. Wilson Reports to the Judges.

At the Science Fair, interest was still centered on Alec Marson's strange machine (which Mr. Wilson had returned to the auditorium).

"Last night," Mr. Wilson told the judges in an excited voice, "I set the selector pointer at # — two strokes down and one across — and pushed the diamond-shaped Change button.

"Nothing showed on the green screen and only the letters A, B, C and D showed on the blue screen. I wrote many different expressions on the green screen — and they simply faded out.

"Once, by accident, I wrote $\frac{12}{2},\frac{3}{1}$ and the machine immediately wrote on the blue screen:

$\frac{12}{2}$, $\frac{3}{1}$	A $\frac{18}{2}$	B $\frac{6}{2}$	C $\frac{36}{2}$	D $\frac{12}{6}$

"My next experiment showed —

	A	B	C	D
$3\overline{)18}, 2\overline{)4}$	$6\overline{)48}$	$6\overline{)24}$	$6\overline{)72}$	$12\overline{)36}$

"I tried another example —

	A	B	C	D
$8\div2, 4\div1$	$16\div2$	$0\div2$	$32\div2$	$8\div8$

The way in which the machine worked was still quite a mystery to me. But as I kept a record of my results, the mystery began to clear up."

The next two pages (15 & 16) explain the way in which the machine works to get such results.

Before reading what he has to say, try to complete the table on this page.

	I write	Machine writes			
		A	B	C	D
a.	$\frac{15}{1}$, $\frac{3}{1}$	$\frac{18}{1}$	$\frac{12}{1}$	$\frac{45}{1}$	$\frac{15}{3}$
b.	$\frac{20}{2}$, $\frac{2}{1}$	$\frac{24}{2}$	$\frac{16}{2}$	$\frac{40}{2}$	$\frac{20}{4}$
c.	$\frac{20}{2}$, $\frac{5}{1}$	$\frac{30}{2}$	$\frac{10}{2}$	$\frac{100}{2}$	$\frac{20}{10}$
d.	$\frac{12}{2}$, $\frac{6}{1}$	$\frac{24}{2}$	$\frac{0}{2}$	$\frac{72}{2}$	$\frac{12}{12}$
e.	$\frac{20}{5}$, $\frac{4}{2}$	$\frac{60}{10}$	$\frac{20}{10}$	$\frac{80}{10}$	$\frac{40}{20}$
f.	$\frac{12}{3}$, $\frac{4}{2}$	$\frac{36}{6}$	$\frac{12}{6}$	$\frac{48}{6}$	$\frac{24}{12}$
g.	$\frac{12}{4}$, $\frac{6}{2}$	$\frac{48}{\quad}$	$\frac{0}{\quad}$	$\frac{72}{\quad}$	$\frac{24}{24}$
h.	$\frac{12}{2}$, $\frac{6}{3}$	$\frac{48}{\quad}$	$\frac{24}{\quad}$	—	$\frac{36}{12}$
i.	$\frac{30}{3}$, $\frac{10}{5}$	$\frac{180}{\quad}$	$\frac{120}{\quad}$	—	$\frac{150}{30}$
j.	$\frac{25}{5}$, $\frac{2}{2}$	$\frac{60}{\quad}$	$\frac{\quad}{10}$	—	$\frac{50}{10}$
k.	$\frac{36}{4}$, $\frac{3}{1}$	$\frac{48}{\quad}$	$\frac{24}{\quad}$	—	$\frac{36}{12}$
l.	$\frac{18}{3}$, $\frac{4}{2}$	—	$\frac{24}{\quad}$	—	$\frac{\quad}{12}$
m.	$\frac{9}{1}$, $\frac{6}{2}$	—	$\frac{12}{\quad}$	—	—
n.	$\frac{4}{2}$, $\frac{4}{2}$	$\frac{16}{4}$	$\frac{0}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{8}$
o.	$\frac{30}{2}$, $\frac{9}{3}$	—	—	—	—
p.	$\frac{30}{3}$, $\frac{25}{5}$	—	—	—	—
q.	$\frac{44}{11}$, $\frac{6}{3}$	—	—	—	—
r.	$\frac{24}{4}$, —	$\frac{108}{\quad}$	$\frac{36}{\quad}$	—	$\frac{72}{\quad}$
s.	$\frac{48}{\quad}$, $\frac{16}{\quad}$	$\frac{\quad}{6}$	$\frac{48}{\quad}$	—	$\frac{\quad}{48}$

"The meaning of A, B, C and D on the blue screen became clear when I rewrote my table of results (shown on page 14) as whole numbers rather than as indicated divisors.

	I write	Machine writes			
		A	B	C	D
a.	15 , 3	18	12	45	5
b.	10 , 2	12	8	20	5
c.	10 , 5		5	50	2
d.	6 , 6			36	1
e.	4 , 2				2
f.	4 , 2				
g.					
h.					
i.					
j.					
k.					
l.					
m.					
n.					
o.					
p.					
q.					
r.					
s.					

"With that mystery solved — A, B, C and D indicating addition, subtraction, multiplication and division, respectively — I wondered how the machine got its particular form of result. It is true that $20 \div 2 \times 10 \div 2 = 200 \div 4$ (see example below) because $10 \times 5 = 50$, but why didn't the machine write $50 \div 1$ as the result; or $100 \div 2$ or $500 \div 10$?

"So, I compared my record with the machine's response in 10 of the examples under column C."

		C
a.	$\frac{15}{1}, \frac{3}{1}$	$\frac{45}{1}$
b.	$\frac{20}{2}, \frac{2}{1}$	$\frac{40}{2}$
c.	$\frac{20}{2}, \frac{5}{1}$	—
d.	$\frac{12}{2}, \frac{6}{1}$	—
e.	$\frac{20}{5}, \frac{4}{2}$	—

		C
f.	— , —	—
g.	— , —	—
h.	— , —	—
i.	— , —	—
j	— , —	—

"One way the machine would work to get these results became obvious immediately, so I turned to the results in column D."

		D
a.	— , —	—
b.	— , —	—
c.	— , —	—
d.	— , —	—
e.	— , —	—

		D
f.	— , —	—
g.	— , —	—
h.	— , —	—
i.	— , —	—
j.	— , —	—

"One way the machine may work is this:"

I write	Machine works		Machine writes
$\frac{20}{2}, \frac{5}{1}$	20 5 2 1	1 x 20 = 20 5 x 2 = 10	$\frac{20}{10}$
$\frac{12}{2}, \frac{6}{3}$	12 6 2 3	3 x 12 = 36 6 x 2 = 12	$\frac{36}{12}$

"The results in the A column gave me more trouble," Mr. Wilson continued. "I made up a pattern or series of examples and noted the following results:"

	I write	Machine writes	I check
A	$\frac{10}{1}, \frac{6}{1}$	$\frac{16}{1}$	10 + 6 = 16
B	$\frac{10}{1}, \frac{6}{2}$	$\frac{26}{2}$	___ + ___ = ___
C	$\frac{10}{1}, \frac{6}{3}$	$\frac{36}{}$	___ + ___ = ___
D	$\frac{10}{2}, \frac{6}{1}$	$\frac{22}{2}$	___ + ___ = ___
E	$\frac{10}{2}, \frac{6}{2}$	$\frac{32}{4}$	___ + ___ = ___
F	$\frac{10}{2}, \frac{6}{3}$	$\frac{42}{}$	___ + ___ = ___
G	$\frac{10}{5}, \frac{6}{1}$	$\frac{40}{}$	___ + ___ = ___
H	$\frac{10}{5}, \frac{6}{1}$	$\frac{50}{10}$	___ + ___ = ___
I	$\frac{10}{5}, \frac{3}{1}$	$\frac{60}{}$	___ + ___ = ___

"I knew that the results in column B were, in some way, based on subtraction, and I made up a similar set of examples." (You can complete the record.)

	I write	Machine writes	I check
a.	$\frac{12}{1}, \frac{6}{2}$	$\frac{18}{2}$	12 − ___ = ___
b.	$\frac{12}{1}, \frac{6}{3}$	$\frac{30}{3}$	___ − ___ = ___
c.	$\frac{12}{2}, \frac{6}{2}$	$\frac{}{4}$	6 − ___ = ___
d.	$\frac{12}{2}, \frac{6}{3}$	$\frac{}{6}$	___ − ___ = ___
e.	$\frac{12}{3}, \frac{6}{2}$	___	___ − ___ = ___
f.	$\frac{12}{3}, \frac{6}{3}$	$\frac{}{9}$	___ − ___ = ___
g.	$\frac{12}{4}, \frac{6}{2}$	___	___ − ___ = ___
h.	$\frac{12}{4}, \frac{6}{3}$	___	___ − ___ = ___
i.	$\frac{12}{4}, \frac{6}{6}$	___	___ − ___ = ___

"I saw a pattern," Mr. Wilson went on, "that seemed to lead me to the same results as the machine. So, I tried more examples. Here are the records I kept:"

	I write	Machine writes A	Machine writes B	I check
I	$\frac{4}{2}, \frac{2}{1}$	$\frac{8}{2}$	$\frac{0}{2}$	A 2 + 2 = 4 B 2 − 2 = 0
II	$\frac{6}{2}, \frac{2}{1}$	___	___	A B
III	$\frac{8}{2}, \frac{2}{1}$	___	___	A B
IV	$\frac{10}{2}, \frac{2}{1}$	___	___	A B
V	$\frac{6}{2}, \frac{3}{1}$	___	___	A B
VI	$\frac{9}{3}, \frac{3}{1}$	___	___	A B
VII	$\frac{20}{5}, \frac{3}{1}$	___	___	A B
VIII	$\frac{12}{2}, \frac{6}{3}$	___	___	A B
IX	$\frac{5}{1}, \frac{6}{3}$	___	___	A B
X	$\frac{10}{5}, \frac{6}{3}$	___	___	A B
XI	$\frac{9}{3}, \frac{4}{2}$	___	___	A B
XII	$\frac{10}{2}, \frac{14}{7}$	___	___	A B

"I'm a little troubled," Judge Harris said, "by this development with Alec's machine. It seems to say that

$$\frac{10}{2} + \frac{6}{2} = \frac{20}{4} + \frac{12}{4} = \frac{32}{4}$$

in example E on page 16. We check by noting that

$$5 + 3 = 8$$

and we know that $32 \div 4 = 8$."

Judge Thornton broke in, "But it seems that Alec invented his machine to work in a way that uses only fractions."

"True," Judge Harris admitted, "but I'm sure Alec knew how to use the least common multiple as the lowest common denominator. Yet his machine seems to say

$$\frac{12}{4} - \frac{6}{6} = \frac{72}{24} - \frac{24}{24} = \frac{48}{24}$$

Yet, I feel quite sure that Alec would have written

$$\frac{12}{4} - \frac{6}{6} = \frac{36}{12} - \frac{12}{12} = \frac{24}{12}$$

In fact, if Alec were here, I'm sure he would simplify each of those fractions and write

$$\frac{12}{4} - \frac{6}{6} = \frac{3}{1} - \frac{1}{1} = \frac{2}{1}$$

Try the following examples in all three ways. I think you will agree with me."

The Machine's Method: Find a common denominator by multiplying the denominators.	Least Common Denominator: Find the least common multiple of the denominators.	Simplify the fractions
$\frac{12}{2} - \frac{6}{2} = \frac{24}{4} - \frac{12}{4} = \frac{\quad}{\quad}$	$\frac{12}{2} - \frac{6}{2} = \frac{\quad}{2} = \frac{\quad}{1}$	$\frac{12}{2} - \frac{6}{2} = \frac{6}{1} - \frac{\quad}{1} = \frac{\quad}{1}$
$\frac{12}{3} - \frac{6}{3} = \frac{\quad}{9} - \frac{\quad}{9} = \frac{\quad}{9}$	$\frac{12}{3} - \frac{6}{3} = \frac{\quad}{3} = \frac{\quad}{1}$	$\frac{12}{3} - \frac{6}{3} = \frac{\quad}{1} - \frac{\quad}{1} = \frac{\quad}{1}$
$\frac{8}{2} + \frac{12}{6} = \frac{\quad}{12} + \frac{\quad}{12} = \frac{\quad}{\quad}$	$\frac{8}{2} + \frac{12}{6} = \frac{\quad}{6} + \frac{\quad}{6} = \frac{\quad}{6} = \frac{\quad}{1}$	$\frac{8}{2} + \frac{12}{6} = \frac{\quad}{1} + \frac{\quad}{1} = \frac{\quad}{1}$
$\frac{20}{10} + \frac{15}{5} = \frac{\quad}{50} + \frac{\quad}{50} = \frac{\quad}{50}$	$\frac{20}{10} + \frac{15}{5} = \frac{\quad}{10} + \frac{\quad}{10} = \frac{\quad}{10} = \frac{\quad}{1}$	$\frac{20}{10} + \frac{15}{5} = \frac{\quad}{1} + \frac{\quad}{1} = \frac{\quad}{1}$
$\frac{4}{4} + \frac{14}{14} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{4}{4} + \frac{14}{14} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{4}{4} + \frac{14}{14} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$
$\frac{24}{6} - \frac{12}{4} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{24}{6} - \frac{12}{4} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{24}{6} - \frac{12}{4} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$
$\frac{24}{12} + \frac{8}{4} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{24}{12} + \frac{8}{4} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{24}{12} + \frac{8}{4} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$
$\frac{6}{3} + \frac{36}{18} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{6}{3} + \frac{36}{18} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{6}{3} + \frac{36}{18} = \frac{\quad}{\quad} + \frac{\quad}{\quad} = \frac{\quad}{\quad}$
$\frac{30}{6} - \frac{16}{8} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{30}{6} - \frac{16}{8} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{30}{6} - \frac{16}{8} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$
$\frac{16}{8} - \frac{12}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{16}{8} - \frac{12}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$	$\frac{16}{8} - \frac{12}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad}$

Mr. Wilson has an idea

"As I was working these examples," Mr. Wilson said, "it occurred to me that we may be unfair to Alec. He never told us what his machine was designed to do. Perhaps we are using his machine in a way he didn't intend it to be used."

In the midst of the discussion of Alec's machine, Mr. Wilson looked up, his eyes wide with a new thought.

"I wonder what kind of project Alec's little sister, Sue, entered in the school fair," he said.

He hurried off, and soon returned with a set of large cards. A neatly printed title was on the top card:

ARITHMETIC GAMES

by Sue Marson

Mr. Wilson and the judges studied Sue's first game. The directions were quite clear.

"Let's play this game," Judge Thornton suggested, and Judge Harris readily agreed.

When they had finished, the game card looked like this:

	Multiply by			Multiply by		
Judge Harris	(7)	(1)		(6)	(1)	Judge Thornton
5	2	3	17	2	5	7
4	4	0	28	4	4	8
6	4	2	30	5	0	5
2	2	0	14	2	2	4
17	12	5	89	13	11	24

"Judge Harris won," Mr. Wilson said, "because his final score of 17 is lower than Judge Thornton's 24."

"That bottom check-row was certainly a good idea on Sue's part," Judge Harris remarked. "It's something like a Cross Number Puzzle."

Look at the completed game. How many of Sue's rules can you guess from seeing these results?

RULE I. Each player writes 1 in the circle at the top of the green column on his side of the board.

RULE II. In the circle at the top of the blue column, each player writes any number he chooses. (Judge Harris selected 7 and Judge Thornton chose 6.) These are the multipliers.

RULE III. The first player (Judge Harris) writes any number he wants in the top box in the center column. (Judge Harris chose 17.)

RULE IV. Both players write in the blue and green columns so that the sum of the products of those numbers and the multipliers taken in pairs is the same as the number in the top box of the center column. (Judge Harris thought:

$2 \times (7) + 3 \times (1) = 14 + 3 = 17$)

(Judge Thornton thought:

$2 \times (6) + 5 \times (1) = 12 + 5 = 17$)

RULE V. Add the last two numbers written in the adjoining space.

$5 = 2 + 3$ and $2 + 5 = 7$

RULE VI. The second player writes any number he chooses in the second box in the center column. (Judge Thornton wrote 28.)

RULE VII. The player with the lowest total in the outside column wins.

Here is a sample game for you to complete:

1st player	(8)	(1)		(5)	(1)	2nd player
4	1	3	11	2	1	3
	3	1	25	5	0	
			9	1		
	3			6	0	

Below are partial records of some games played according to Sue Marson's rules. Complete the records.

I

3	1		9	1	
9	8	1	25		9
			18		
	7	0			
				8	9

II

○	1		○	1	
	3	1	13	1	6
			39		
			26		
9		1			
33					

III

9	1		5	1	
		45			
		35			
1	1				
				3	
			21		

IV

○	1		○	1	
10	8		50		2	8
				3	1	
	5	3				
	21	11				

Another of Sue's games was identical with the first, except that in it each player has an extra play.

V

7	1		○	1	
			1	4	
	2	1			5
8		6	3		5
		25			
			2		3
		114			

VI

○	1		4	1	
			29		
8		1			
6	6		18		
			27		
7	6				
	40	6			

Another of Sue Marson's Games.

1. Follow rules given for the last game on page 19.
2. Players take turns filling in any numbers in the center column until it is completed.
3. Each player spins the dial at right.
4. Fill in top row from (1) to the left by multiplying number in previous circle by numbers selected on the dial.

Reconstruct these three games:

	○	○	8	4	2	1		○	○	27	9	3	1	
3	0	0	1	0	1	1	11	0	0	0	1	0	2	3
3							38							4
							26							
							63							

	2,401	343	49	7	1					5	1	
						67						
						700						
						153						
						2,500						

	6,561			9	1		10,000			10	1	
						75						
							0	0	2	6		9
						6,571						
						13,122						

Mr. Wilson asked Mrs. Esser to have her class try some of Sue Marson's games. The rules for the first were:

Rule I: Multiply each number in the green-tinted row by the number at the top of each column. Add the products. This sum must be the same as the number in the untinted box on the left or right.

Rule II: Fill in the row tinted blue with numbers (following Rule I) so that the sum of these numbers is as small as possible.

25 5 1	36 6 1	◯ 4 1	10 1
42 0 7	42 12	42 2 1 6	3 12
1 3			
27 9 3 1	49 7 1	9 1	10 1
64 1 2 4	64 0 8	64 19	24
			6 4
5 1	8 1	16 8 4 2 1	10 1
8 3	23 1	23 0 0 0 3	23 1
4 3			
16 4 1	7 1	27 9 3 1	10 1
45 0 9	45 5	45 5	25
			4 5
49 7 1	64 8 1	81 9 1	10 1
95 1 3	95 1 15	95 0 9	6
			9 5
64 16 4 1	36 6 1	9 1	10 1
89 0 1 5	89 1 17	89 7	4
			8 9

Another of the Games.

Rule I: All numbers in tinted boxes must follow Rule I of the previous game and, in rows 1, 2 and 4, their sums must be as small as possible.

Rule II: Numbers in 3rd row are to be the sum of the two numbers written above it.

Rule III: Numbers in blue-tinted boxes must express the same number as in 3rd row, using smallest numbers.

A.

8	4	2	1		16	4	1
1	0	1	1	11	0	2	3
	0	1	1	3	0	0	
		2	2		0		
			0				

B.

32	16	4	1		25	5	1
			2	38	1		
			3	15	0		
			5		1		
			1		2		

C.

125	25	5	1		100	10	1
0	4	4	4				
					0	8	7

D.

125	25	5	1		100	10	1
2	0	3	3				
3	1	7	6				

E.

125	25	5	1		100	10	1
					0	7	7
					0	15	16

F.

125	25	5	1		100	10	1
0	3	2	2				
0	5	4	6				

G.

125	25	5	1		100	10	1
				223			
4	3	5	7				

H.

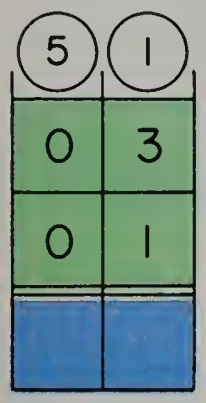
125	25	5	1		100	10	1
1	2	2	0				
					3	10	10

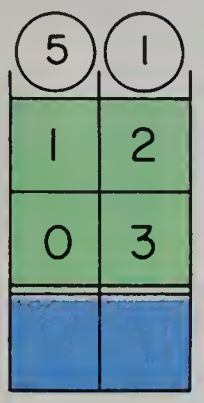
Another Game Sue Invented.

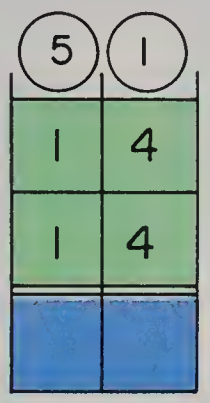
Rule I: Multiply the number in each box in a row by the number at the top of the column. The sum of the products is the number expressed by that row (as in other games).

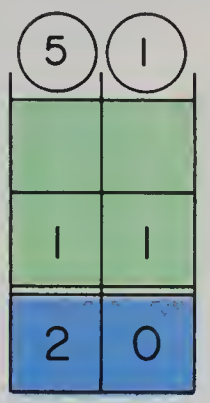
Rule II: The number expressed by the bottom row is to be the sum of the numbers expressed by the rows above it.

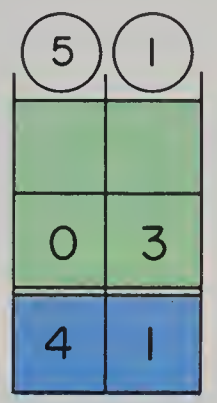
Rule III: Only 0, 1, 2, 3 or 4 should be written in any one box.

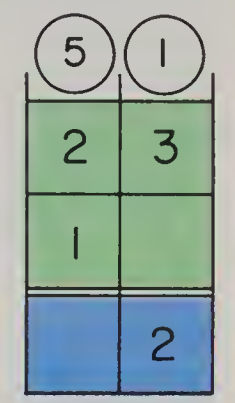


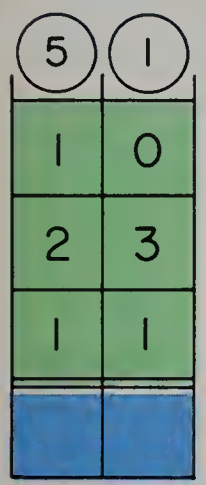


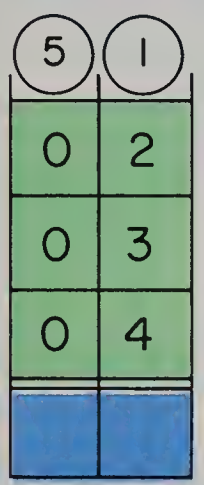


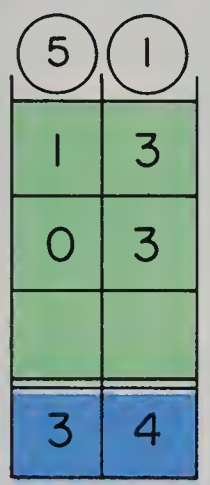


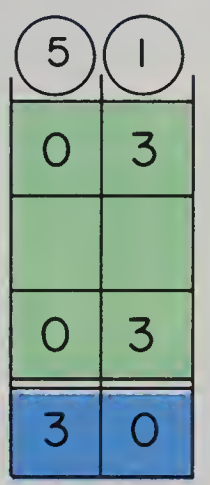


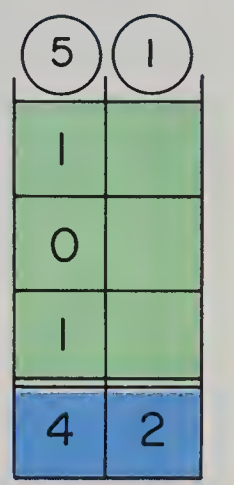


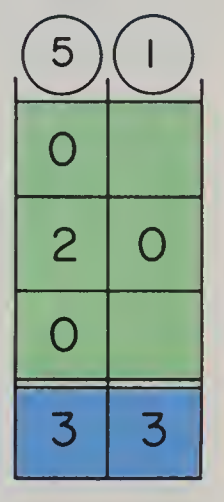


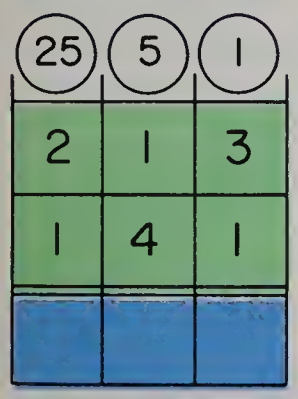


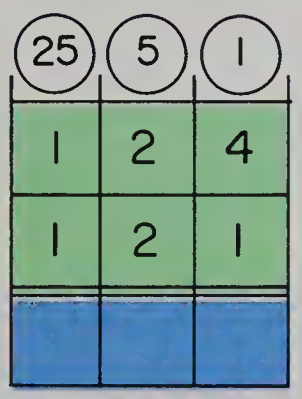


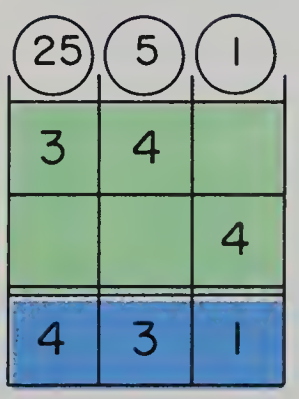


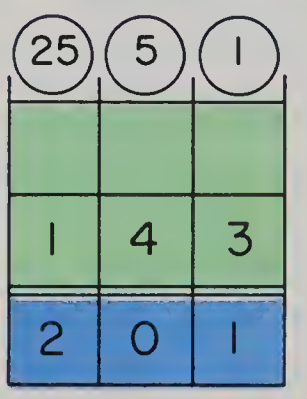


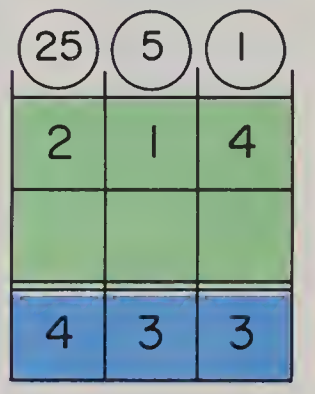


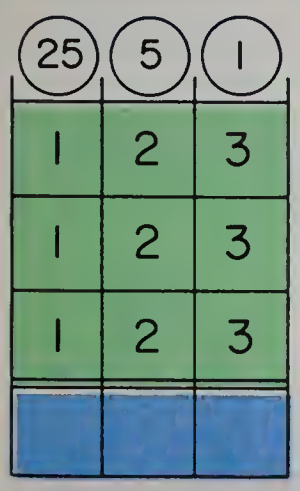


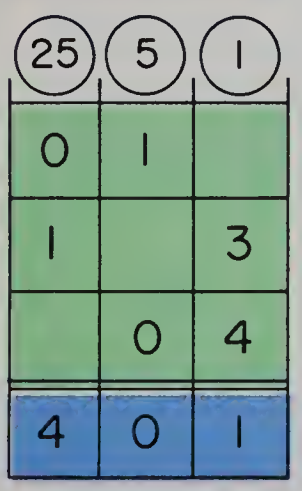


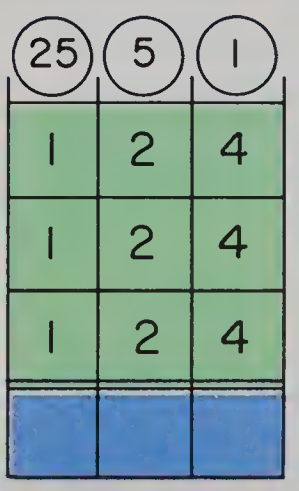


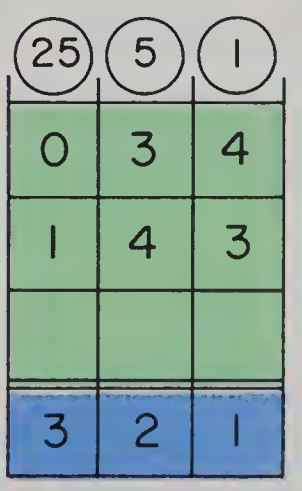


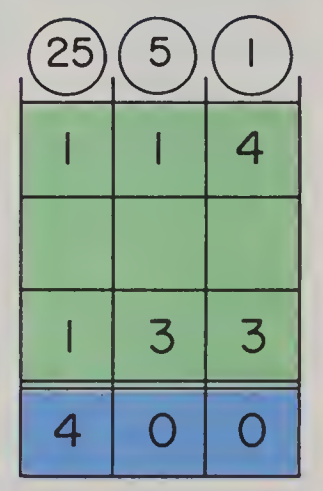


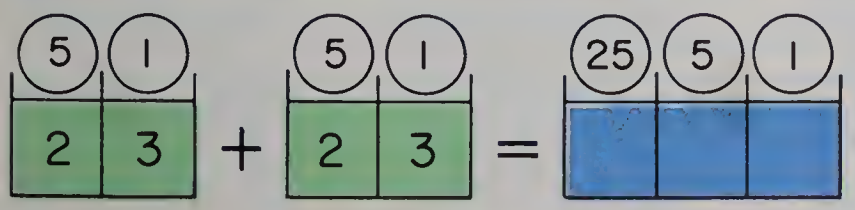


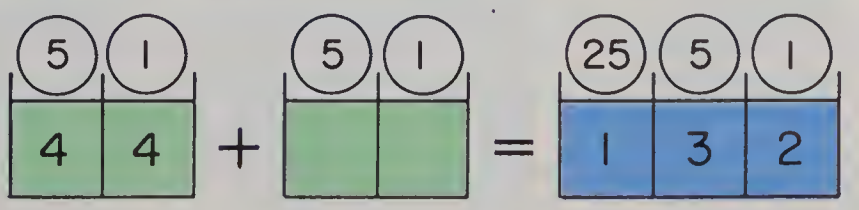












In another game, Sue expands the idea behind the game explained on the previous page. She changes rule III.

New Rule III: In each example, all numbers must be less than the one in the blue circle.

5	1
0	4
2	3

25	5	1
1	4	2
4	0	0

9	3	1
0	2	1
1	2	2

9	3	1
0	1	2
0	1	2

9	3	1
0	1	2
2	0	0

2	1
0	1
0	1

4	2	1
0	1	1
0	0	1

4	2	1
0	1	1
1	1	0

16	4	1
1	2	3
1	2	3

16	4	1
2	1	3
3	0	0

6	1
1	3
2	4

36	6	1
1	2	3
2	3	4

36	6	1
1	3	
1		5
	1	4

49	7	1
1	2	3
0	5	4

49	7	1
2	3	4
	6	5
4		

5	1
1	3
1	
	4
4	1

25	5	1
1	3	
0	3	
1	3	
	1	2

64	8	1
1	2	3
4	5	6
0	6	7

81	9	1
1	7	8
3	7	8
8	6	5

81	9	1
0	3	
1		7
	5	8
4	5	3

10	1
3	5
1	7
0	9

100	10	1
1	3	
2		8
	7	9
8	7	3

100	10	1
1	5	9
2	5	9
	5	9
7		

100	10	1
1	8	6
1	8	6
1	8	6

100	10	1
2	7	4
1		9
	8	
9	3	1

Can you work out the rules for these games from the examples given?

Game A

a. 2 x

25	5	1
0	3	2

=

25	5	1
1	1	4

b. 3 x

25	5	1
1	1	3

=

25	5	1
3		4

c. 2 x

9	3	1
0	1	2

=

9	3	1
		1

d. 5 x

49	7	1
1	2	5

=

49	7	1

e. 4 x

100	10	1
1	3	3

=

100	10	1

f. 3 x

25	5	1
0	4	3

=

25	5	1

g. 7 x

100	10	1
1	1	5

=

100	10	1

h. 6 x

81	9	1
0	3	2

=

81	9	1

i. 9 x

100	10	1
0	1	7

=

100	10	1

j. 8 x

100	10	1
1	2	4

=

100	10	1

Game B

k.

25	5	1
4	2	4

÷ 2 =

25	5	1
2	1	2

l.

25	5	1
3	1	4

÷ 3 =

25	5	1
1	0	

m.

36	6	1
5	3	2

÷ 5 =

36	6	1
	0	

n.

100	10	1
1	0	4

÷ 4 =

100	10	1

o.

25	5	1
4	0	4

÷ 4 =

25	5	1

p.

100	10	1
1	3	2

÷ 3 =

100	10	1

q.

25	5	1
3	0	1

÷ 4 =

25	5	1

r.

36	6	1
2	0	4

÷ 4 =

36	6	1

s.

100	10	1
0	7	6

÷ 4 =

100	10	1

t.

100	10	1
4	0	6

÷ 7 =

100	10	1

One day, Mrs. Esser said to her arithmetic class: "I've thought of a game like Sue's. I call it

FEWEST COINS AND BILLS

a. Blue: amount of purchase
b. Green: amount given the clerk
c. White: amount of change

"On these pages, there are incomplete records of sales made in a store.

"In the name I have given the game, there is a clue to the way in which the records are to be completed."

	Ten dollar bills	Five dollar bills	One dollar bills	Half dollars	Quarters	Dimes	Nickels	Pennies
Six dollars and thirty-two cents	0	1	1	0	1	0	1	2
Seven dollars	0	1	2	0	0	0	0	0
Sixty-eight cents								
Forty-one dollars and twenty-three cents								
Forty-five dollars	4	1	0	0	0	0	0	0
Two dollars and eleven cents								
Five dollars								
Thirty-five dollars								
Two dollars and twenty-seven cents								
Eighty-two dollars and thirty-one cents								
Eighty-five dollars and fifty cents								
	7	1	2	0	1	0	0	3
	9	0	0	0	0	0	0	0
Nineteen dollars and ninety-one cents								
Twenty dollars								

"Now let's change the rules a bit," Mrs. Esser said.

New Rule: The only coins and bills to be used are pennies, dimes, one dollar bills, ten dollar bills and one hundred dollar bills.

"And, instead of writing the amount in words, we will use shorthand."

	Money Shorthand	One hundred dollar bills	Ten dollar bills	One dollar bills	Dimes	Pennies
I	$.93					
	$5.00					
II		0	5	7	6	3
		0	6	0	0	0
III						
		1	3	0	0	0
		0	0	4	7	5
IV						
	$30.00					
		0	0	8	6	3
V		1	9	0	8	7
		2	0	0	0	0
VI						
		2	0	0	5	0
		1	2	4	3	3
VII	$18.84					
	$101.00					
VIII						
		2	0	0	0	0
		0	0	8	9	5
IX		0	1	3	8	4
		0	0	7	0	6
X		3	7	1	3	6
	$580.00					
XI						
		7	0	1	5	1
		0	2	0	2	2
XII	$66.69					
		0	2	9	3	1
XIII		7	5	5	1	8
	$800.50					
XIV						
		9	0	1	0	0
	$64.46					
XV		4	0	7	8	3
		5	1	0	0	0

"This kind of game," Mrs. Esser explained, "needn't be limited to amounts of money. Here are rules and examples involving ways to express the same amounts of time.

Rule I: Blue and green rows between double-black lines express the same amount of time.

Rule II: Blue row always uses smallest possible total of units."

A

Weeks	Days
0	8
1	1

B

Weeks	Days
3	21

C

Weeks	Days
17	17

D

Weeks	Days
6	6

E

Days	Hours
1	24
	0

F

Days	Hours
0	49

G

Days	Hours
10	
13	3

H

Days	Hours
7	
8	9

I

Hours	Min.
2	80
3	

J

Hours	Min.
0	180

K

Hours	Min.
5	
7	17

L

Hours	Min.
	617
10	17

M

Min.	Sec.
0	185

N

Min.	Sec.
13	47

O

Min.	Sec.
	637
12	37

P

Min.	Sec.
9	330

Q

Min.	Sec.
1	300

R

Min.	Sec.
3	195

S

Min.	Sec.
2	
2	58

T

Min.	Sec.
12	158

a

Hours	Min.	Sec.
0.	65	60

b

Hours	Min.	Sec.
10	240	98

c

Hours	Min.	Sec.
7		483
8	0	3

d

Weeks	Days	Hours
1	7	26

e

Weeks	Days	Hours
19	3	100

f

Weeks	Days	Hours
2	70	
12	3	0

g

Days	Hours	Min.
5	11	100

h

Days	Hours	Min.
1	48	600

i

Days	Hours	Min.
12	39	
13	15	39

j

Hours	Min.	Sec.
11	120	87

k

Hours	Min.	Sec.
9	90	99

l

Hours	Min.	Sec.
7		665
8	1	5

m

Weeks	Days	Hours
3	15	50

n

Weeks	Days	Hours
7	7	70

o

Days	Hours	Min.	Sec.
0	30	59	125

p

Days	Hours	Min.	Sec.
4	26	69	87

q

Weeks	Days	Hours	Min.	Sec.
1	11	24	90	125

r

Weeks	Days	Hours	Min.	Sec.
0	9	19	600	128

Mrs. Esser put the following examples on the board and asked the class: "Can you discover sensible rules for these games?"

1.

Weeks	Days
3	4
1	3
	0

2.

Days	Hrs.
2	20
2	5
	1

3.

Hrs.	Min.
0	37
7	60

4.

Min.	Sec.
17	37
9	90

5.

Min.	Sec.
4	65
4	65

6.

Hrs.	Min.
3	25
1	
5	17

7.

Days	Hrs.
8	
1	36
11	12

8.

Min.	Sec.
1	81
2	38

9.

Hrs.	Min.
3	121
	61
10	

10.

Min.	Sec.
7	47
19	
28	47

11.

Min.	Sec.
5	60
7	
17	0

12.

Weeks	Days	Hrs.
1	6	19
1	6	19

13.

Hrs.	Min.	Sec.
8	39	19
19	84	103

14.

Hrs.	Min.	Sec.
2	58	100
4	59	100

15.

Hrs.	Min.
8	35
14	15
19	27

16.

Hrs.	Min.	Sec.
8	42	19
1	58	46
7	37	38

17.

Days	Hrs.	Min.
4	18	50
3	14	28
5	9	57

18.

Hrs.	Min.	Sec.
0	81	121
2	37	66
19	46	59

19. 2 X

Weeks	Days	Hrs.	Min.	Sec.
1	4	3	25	50

=

Weeks	Days	Hrs.	Min.	Sec.
3	1	6		40

20. 3 X

Weeks	Days	Hrs.	Min.	Sec.
0	2	8	5	25

=

Weeks	Days	Hrs.	Min.	Sec.
				15

21. 2 X

Weeks	Days	Hrs.	Min.	Sec.
5	5	15	30	19

=

Weeks	Days	Hrs.	Min.	Sec.

"Here is another game," Mrs. Esser explained, "without any rules given. Can you find rules that make sense of the results shown? Use your rules to complete the examples."

A

	Tens of Dollars	Dollars	Dimes	Pennies				
	0	7	1	3	0	7	1	3
2	0	14	2	6				
7							9	1
9				27				
5				15	3	5	6	5
3	0	21						
8								
10	0	70						
4				12				
6			6					

C

	0	6	8	4
10	6	8	4	0
6				
4	2	7	3	6
7				
2				
9				
5				
3				
8				

B

	0	5	9	2
3		15		
8				16
5				
9				
2			18	
7				
4				
6				
10				20

D

	0	1	6	5
2	0	3	3	0
10				
12				
6				
5				
11				
16				
9				
7				

E

	Tens of Dollars	Dollars	Dimes	Pennies				
	0	4	1	3	0	4	1	3
8		32	8	24	3	3	0	4
3		12	3	9				
11				33				
14								
5								
9								
6								
10								
16								

F

	0	1	9	4
9			4	6
3				2
12				
15				
5				
20				
7				
10				
17				

G

	0	3	2	5
10	3	2	5	0
20	6			
30				
7				
5				
12				
17				
9				
8				
2				
11				
13				

H

	0	2	3	6
10			6	0
5				
15				
20				
9				
11				
40				
2				
42				
8				
30				
38				

I

	0	0	7	8
6				
10				
16				
7				
20				
27				
5				
30				
35				
9				
40				
49				

Mrs. Esser said, "I call this game

SMALLEST TOTAL OF UNITS

Can you discover the rules and put them to work?"

	Weeks	Days	Hours	Yards	Feet	Inches	Gallons	Quarts	Pints	Tens of Dollars	Dollars	Dimes	Pennies
	0	1	8	0	2	7	0	0	1	0	2	9	6
2	0	2	16	0	4	14	0	0	2				
	0	2	16			2	0	1	0				
3	0	3	24								6	27	18
	0	4	0										8
5				0	10	35							
				4	0	11							
4													
6													
10													
7													
8							0	0	8				
							1	0	0				
15													
9													
24													

	Hours	Minutes	Seconds	[illegible]	[illegible]	Tens of Dollars	Dollars	Dimes	Pennies	[illegible]	Hundredths	Tenths	Units
	0	10	12	0	5	0	4	2	8	0	3	7	9
3	0	30	36										
	0	30	36										
5	0	50	60										
	0	51	0										
8				0	40								
				2	8								
2						0	8	4	16				
						0	8	5	6				
7										0	21	49	63
													3
9	0	90	108										
	1	31	48										
4													
6													
10													
16													
14													
19													

How rapidly and accurately can you complete the examples in the green-tinted areas?

3 x 2 =	5 x 6 =	7 x 8 =
3 x 10 =	5 x 10 =	7 x 10 =
3 x 12 =	5 x 16 =	7 x 18 =
8 x 2 =	2 x 9 =	5 x 7 =
8 x 10 =	2 x 10 =	5 x 20 =
8 x 12 =	2 x 19 =	5 x 27 =

3 x4	10 x4	13 x4	8 x6	10 x6	18 x6	2 x9	10 x9	12 x9
5 x3	20 x3	25 x3	4 x8	20 x8	24 x8	6 x7	20 x7	26 x7
7 x2	30 x2	37 x2	8 x4	50 x4	58 x4	5 x9	60 x9	65 x9
9 x6	40 x6	49 x6	5 x5	70 x5	75 x5	7 x8	30 x8	37 x8

3 x 9 =	8 x = 40	9 x 4 =
3 x 50 =	8 x = 480	9 x = 180
3 x 59 =	8 x = 520	9 x = 216
7 x 3 =	6 x 7 =	4 x = 28
7 x 70 =	6 x 60 =	4 x = 280
7 x 73 =	6 x 67 =	4 x = 308

3 x 6	50 x 6	53 x 6	4 x 7	50 x 7	54 x 7	3 x 9	40 x 9	43 x 9
9 x 4	60 x 4	69 x 4	6 x 6	40 x 6	46 x 6	5 x 7	60 x 7	65 x 7
4 x 5	40 x 5	44 x 5	9 x 5	80 x 5	89 x 5	8 x 9	70 x 9	78 x 9
6 x 4	90 x 4	96 x 4	9 x 8	70 x 8	79 x 8	7 x 7	90 x 7	97 x 7
9 x 9	90 x 9	99 x 9	5 x 4	80 x 4	85 x 4	6 x 8	90 x 8	96 x 8

8 x 2	70 x 2	78 x 2	3 x 8	80 x 8	83 x 8	5 x 6	80 x 6	85 x 6
8 x 8	50 x 8	58 x 8	6 x 3	80 x 3	86 x 3	7 x 9	50 x 9	57 x 9
4 x 6	70 x 6	74 x 6	3 x 5	90 x 5	93 x 5	8 x 5	50 x 5	58 x 5
7 x 3	90 x 3	97 x 3	6 x 9	80 x 9	86 x 9	8 x 3	60 x 3	68 x 3
4 x 4	60 x 4	64 x 4	2 x 6	90 x 6	92 x 6	9 x 7	80 x 7	89 x 7

a.	b.	c.
3 x 2 =	6 x 3 =	4 x 5 =
3 x 10 =	6 x 20 =	4 x 20 =
3 x 300 =	6 x 100 =	4 x 300 =
3 x 312 =	6 x 123 =	4 x 325 =

d.	5 x 5	20 x 5	300 x 5	325 x 5	l.	7 x 9	00 x 9	400 x 9	407 x 9
e.	7 x 2	10 x 2	500 x 2	517 x 2	m.	0 x 7	90 x 7	800 x 7	890 x 7
f.	8 x 5	40 x 5	700 x 5	748 x 5	n.	9 x 6	40 x 6	700 x 6	749 x 6
g.	6 x 4	40 x 4	800 x 4	846 x 4	o.	1 x 8	90 x 8	200 x 8	291 x 8
h.	8 x 6	50 x 6	100 x 6	158 x 6	p.	4 x 7	00 x 7	600 x 7	604 x 7
i.	9 x 3	40 x 3	800 x 3	849 x 3	q.	7 x 8	40 x 8	600 x 8	647 x 8
j.	6 x 5	50 x 5	200 x 5	256 x 5	r.	0 x 9	10 x 9	200 x 9	210 x 9
k.	8 x 8	50 x 8	300 x 8	358 x 8	s.	9 x 9	50 x 9	300 x 9	359 x 9

2 x 14 = 28	5 x 16 =	3 x 21 =
10 x 14 = 140	10 x 16 =	20 x 21 =
12 x 14 =	15 x 16 =	23 x 21 =
4 x 12 =	6 x 15 =	1 x 23 =
10 x 12 =	10 x 15 =	20 x 23 =
14 x 12 =	16 x 15 =	21 x 23 =
42 x 9 42 x 10 42 x 19	2 x 19 = 40 x 19 = 42 x 19 =	48 x 7 48 x 20 48 x 27
34 x 5 34 x 20 34 x 25	4 x 25 = 30 x 25 = 34 x 25 =	8 x 27 = 40 x 27 = 48 x 27 =
62 x 5 62 x 30 62 x 35	2 x 35 = 60 x 35 = 62 x 35 =	18 x 3 18 x 20 18 x 23

23 x 18

184 (8 x 23)

230 (10 x 23)

4

18 x 23

(x 18)

(x 18)

324 x 14

(x 324)

(x)

49 x 35

(x 49)

(x 49)

35 x 49

(x)

(x)

152 x 32

(x)

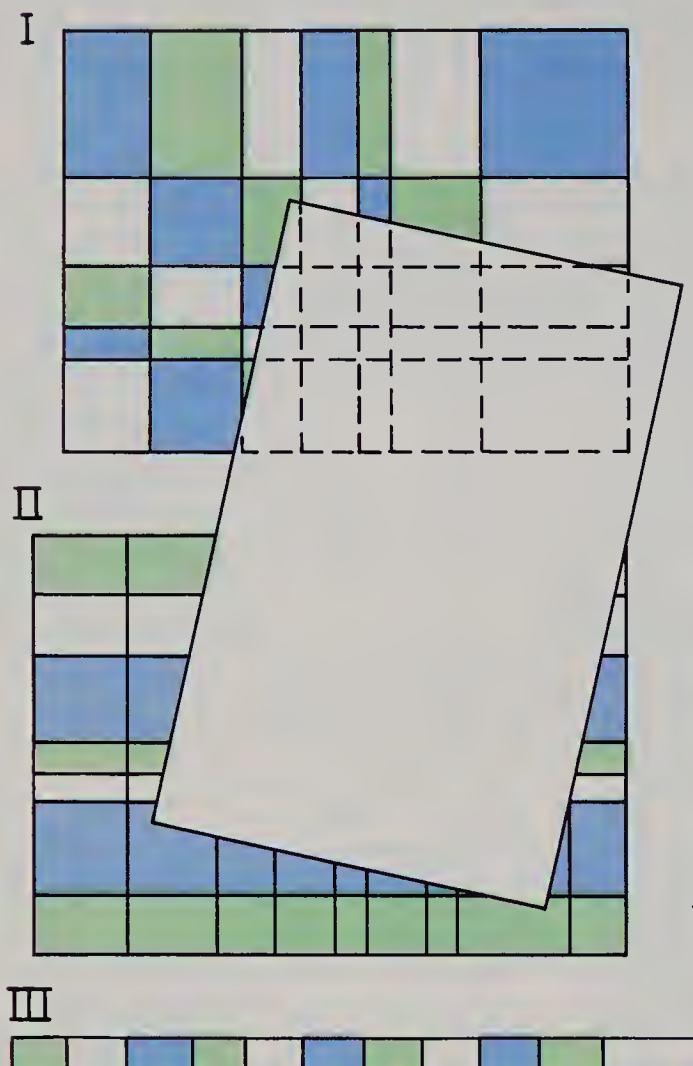
(x)

In the drawings below, cards have been placed so they hide parts of the design.

IF the black lines were extended in straight lines under the card; and

IF the pattern of coloring is continued under the card;

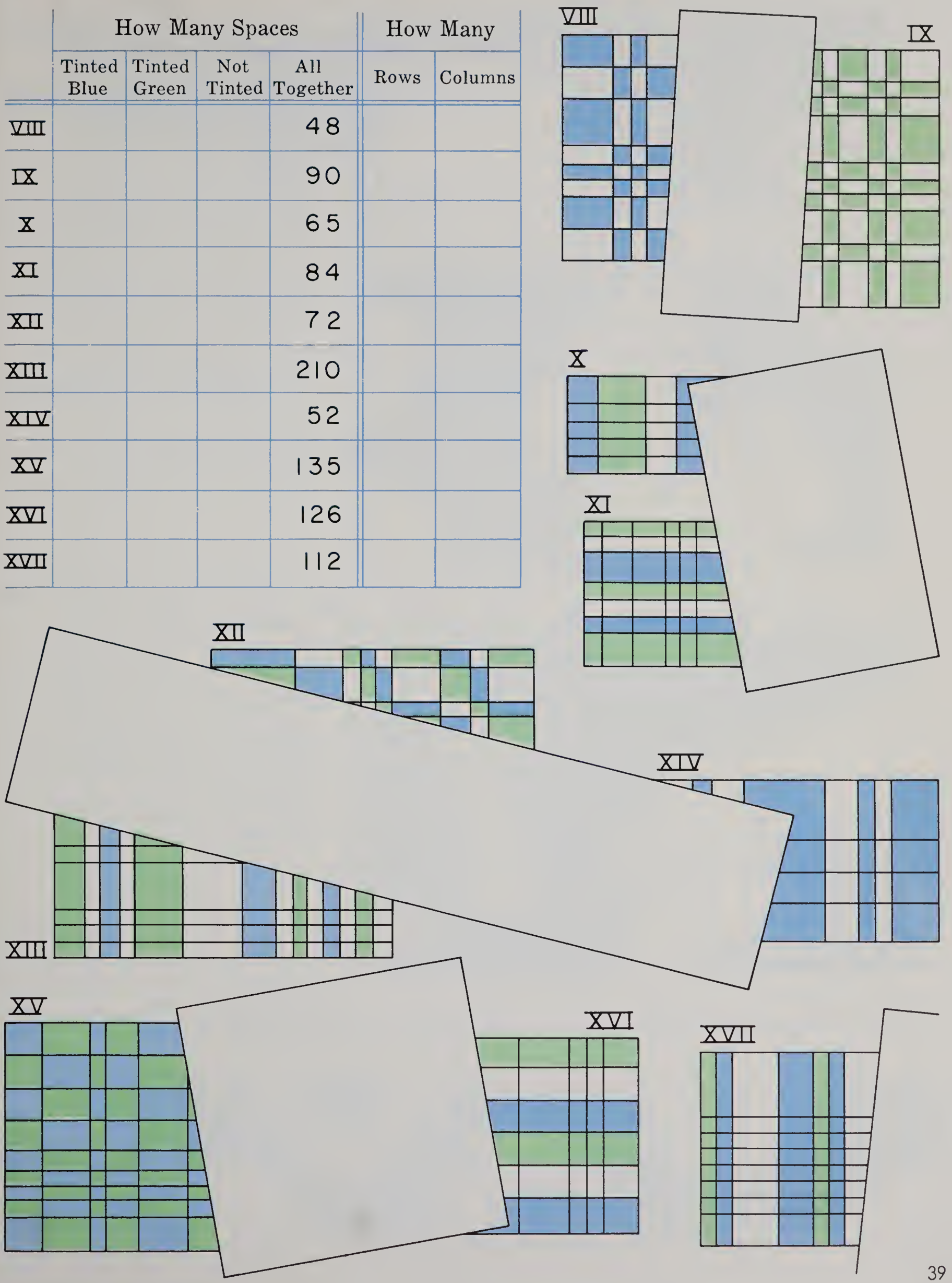
THEN you can complete the following tables concerned with the whole design.

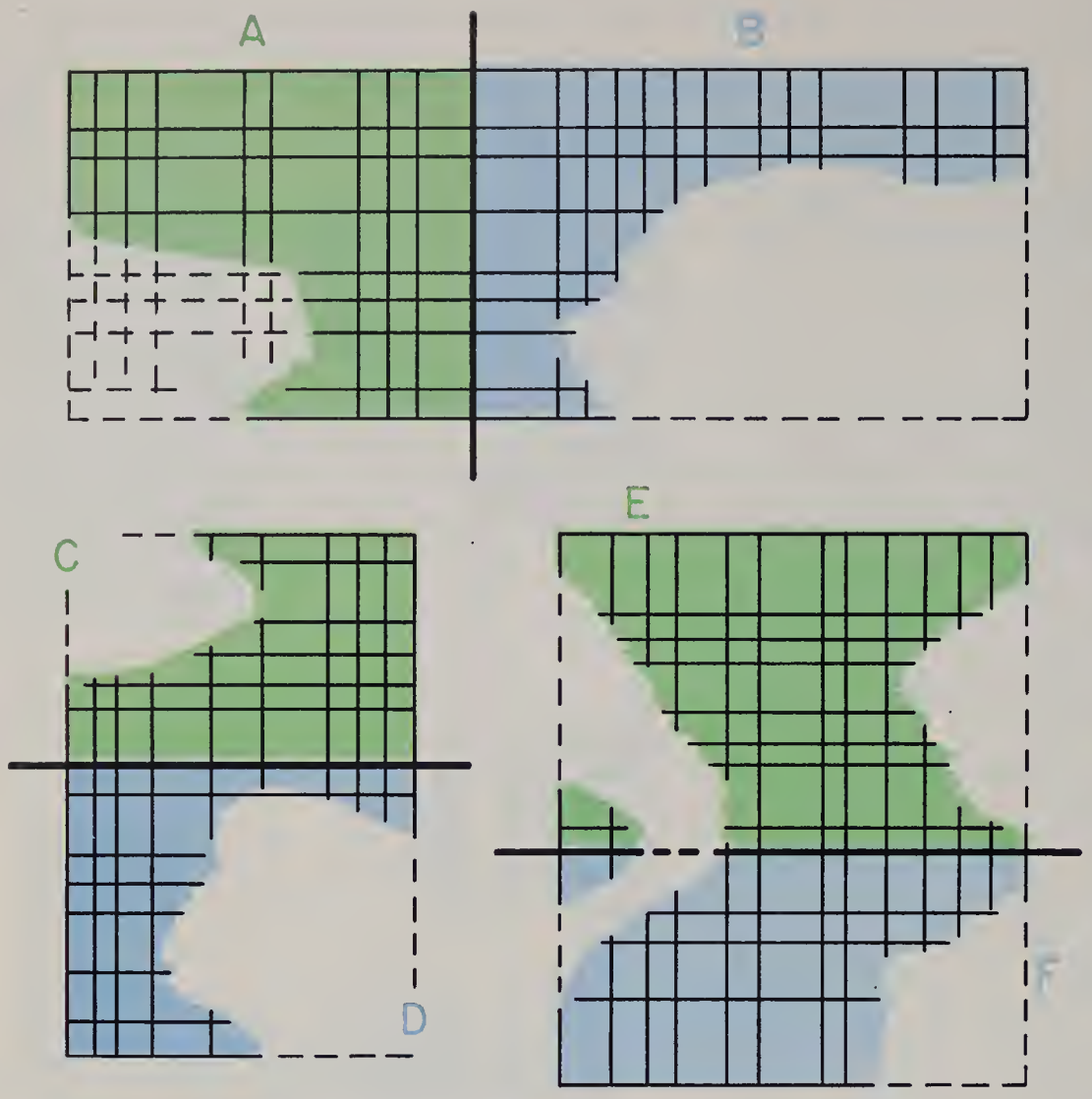


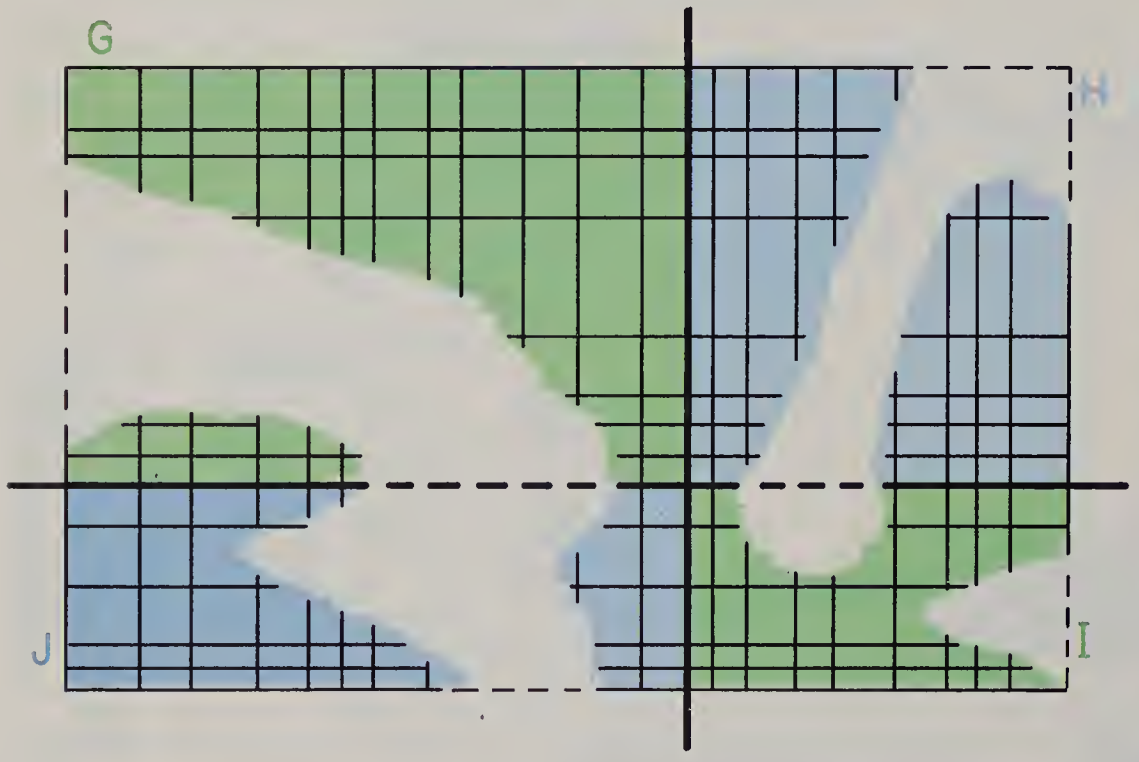
	How Many Spaces				How Many	
	Tinted Blue	Tinted Green	Not Tinted	All Together	Rows	Columns
I						
II						
III						
IV						
V						
VI						
VII						

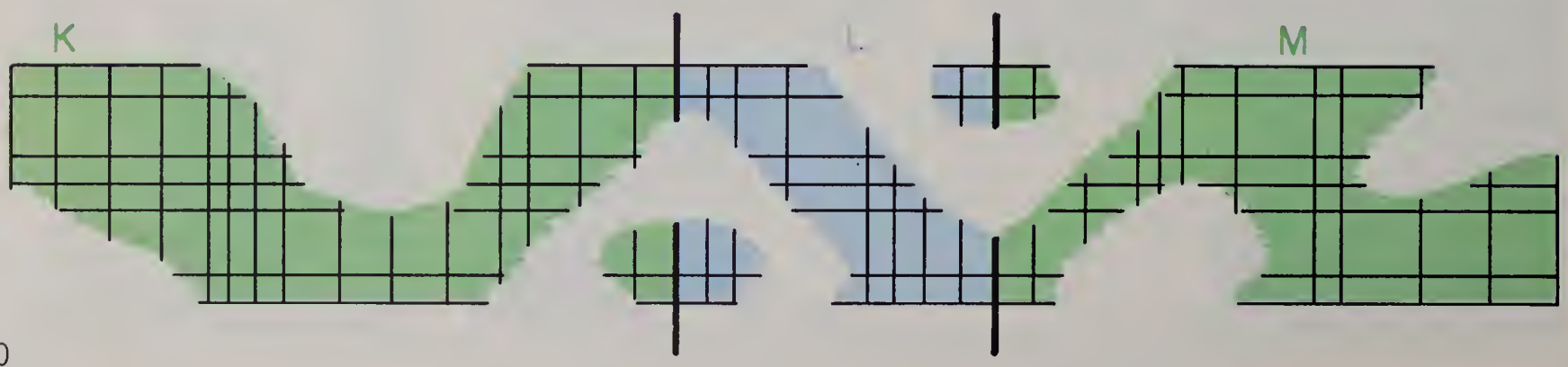
Same rules as page 38.

	How Many Spaces				How Many	
	Tinted Blue	Tinted Green	Not Tinted	All Together	Rows	Columns
VIII				48		
IX				90		
X				65		
XI				84		
XII				72		
XIII				210		
XIV				52		
XV				135		
XVI				126		
XVII				112		

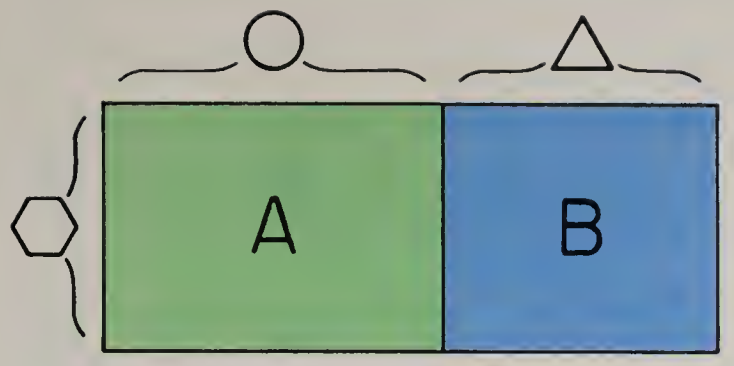








	Number of:		
	Rows	Columns	Spaces
A	8	9	
B	8		
A & B Together	8		
C			
D			
C & D			
E			
F			
E & F			
G			
H			
I			
J			
G & H			
J & I			
G & J			
H & I			
G, H, I & J			
K			
L			
M			
K, L & M			



Any number written in ⬡ indicates the number of rows in A and B.

Any number written in ○ indicates the number of columns in A.

Any number written in △ indicates the number of columns in B.

If the following information in any one line were true, then you can complete that line.

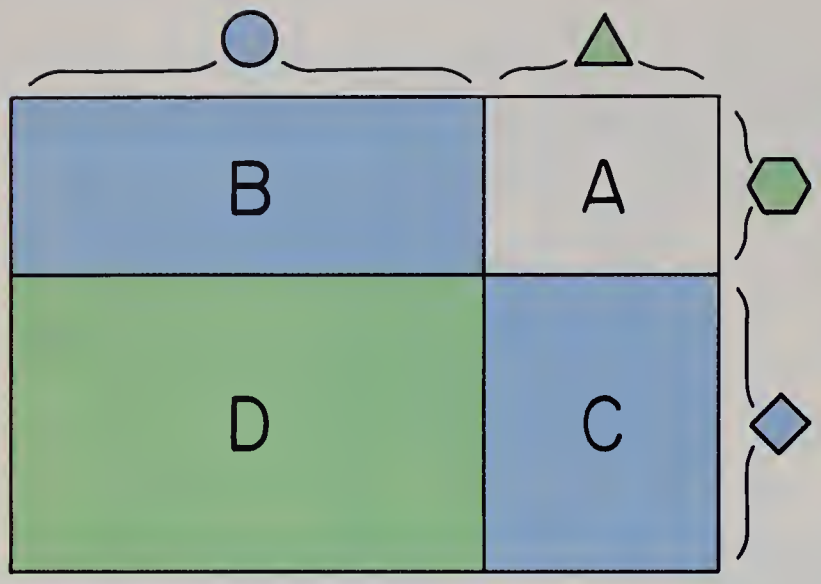
	⬡	○	△	Number of Spaces A	B	A&B
I	3	6	5	18		
II	5			20	25	
III	4	10	7			
IV	8		3			104
V	6				24	144

Rules in following examples:

1. Only multiples of 10 can be written in ○ ;
2. Only numbers less than 10 can be written in △ .

	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV
○	10		30			50			30	
△			2			3			5	7
○ & △	13									
⬡	4	5		8	7		8	9		
B							24			
A	40						320			
A & B	52	105	224	248	133	212		378	210	189

	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV
○ & △						64				
⬡	7	10		7	9		11	4		5
B	21		32						16	
A	70		80						560	
A & B		340		245	324	384	253	348		465



Additional Rules:

Numbers written in ○ and ◇ must be multiples of 10.

Numbers written in △ and ⬡ must be less than 10.

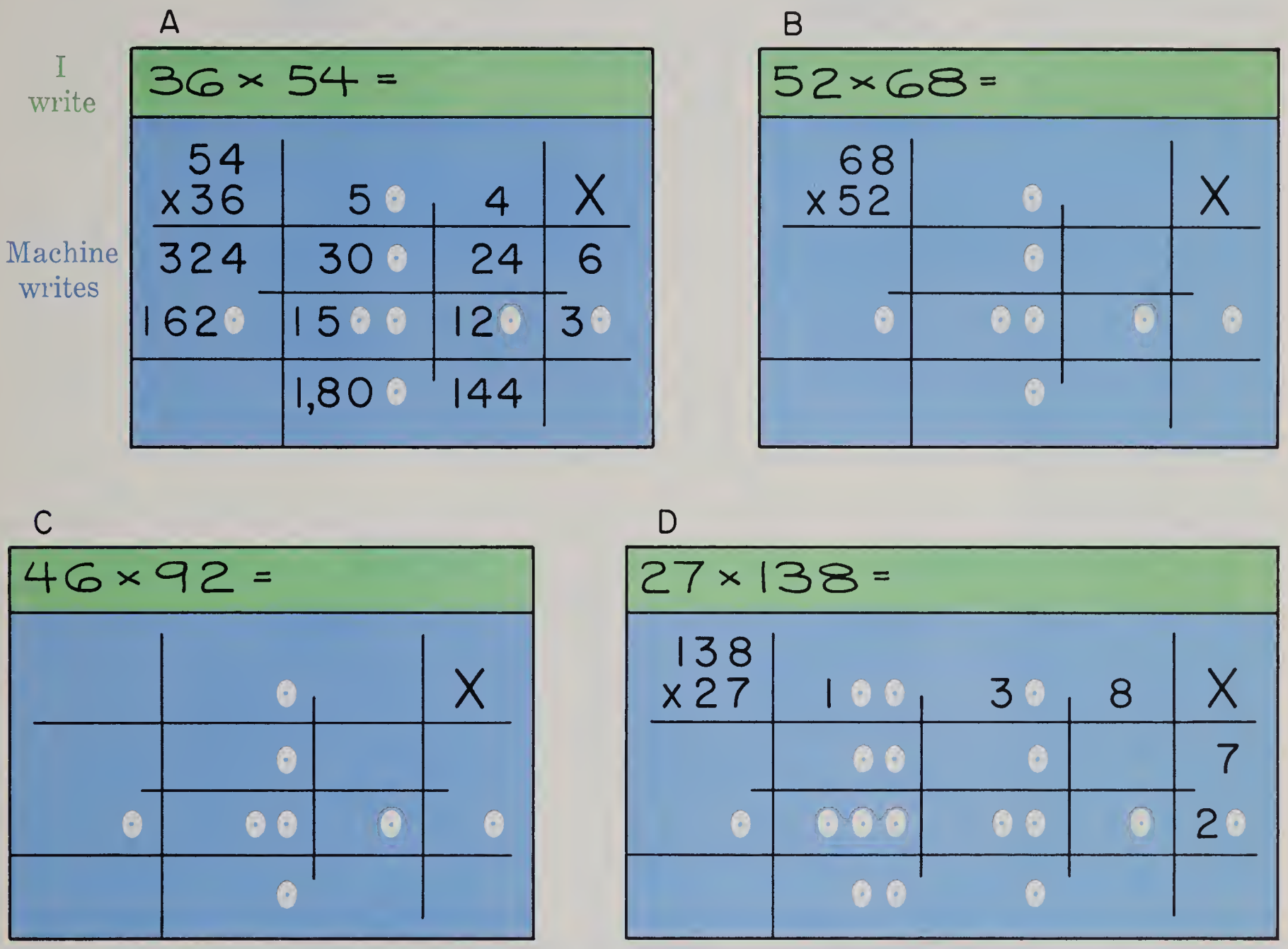
Otherwise, follow the same general rules used on page 41 to complete the records given below.

	I	II	III	IV	V	VI	VII	VIII
○	20	10	20	10	20		70	50
△	3	9	6	6	5		1	5
◇	10	20		20		10	20	
⬡	3	1		3		7	6	
A	9					28		45
B						210		
C								150
D	200							
○ & △								
◇ & ⬡								
A & B								
C & D								
✳ A,B,C & D			312		625			
A & C								
B & D								
✳ Sum of 2 lines above								

Why are the lines marked ✳ identical?

Mr. Wilson, the principal, was visiting the class when they were working on page 42. He asked Mrs. Esser for an opportunity to explain an interesting development.

"You know," he began, "when I experimented with Alec Marson's machine, one of the change buttons produced a result similar to that found on page 42. Here are some of the records I kept of the experiments. Can you fill in the blanks?"



The machine is, of course, simply carrying out such indicated multiplications as:

36 x 54	52 x 68	46 x 92	216 x 23	375 x 41
0	0	0	0	0

Because 0 always occurs where they are shown in blue, it is customary to omit them altogether.

Multiples of 10 in blue-tinted spaces: Numbers less than 10 in green spaces:

39 × 12

	0	9
	0	2
0	00	0 10
468	0	

27 × 27

	20	7
	0	
0	00	0 0
1,161	0	

34 × 34

	0	
	0	4
0	00	0 30
714	0	

19 × 51

	10	9
	0	
0	00	0 0
969	0	

36 × 58

	30	6
	0	
0	00	0 0
2,088	0	

24 × 92

	0	
	0	4
0	00	0 20
2,208	0	

86 × 57

	0	
	0	6
0	00	0 80
4,902	0	

94 × 36

	90	4
	0	
0	00	0 0
3,384	0	

112 × 34

	110	2
	0	
0	00	0 0
3,808	0	

47 × 37

	40	7
	0	
0	00	0 0
1,739	0	

Complete

37 × 23

0

851

45 × 67

0

3,015

309 × 28

0

8,652

239 × 35

8,365

28 × 38

1,064

67 × 76

5,092

If a product and one factor is known, you can discover quite a bit about the missing factors. (Leave green-tinted area until last.)

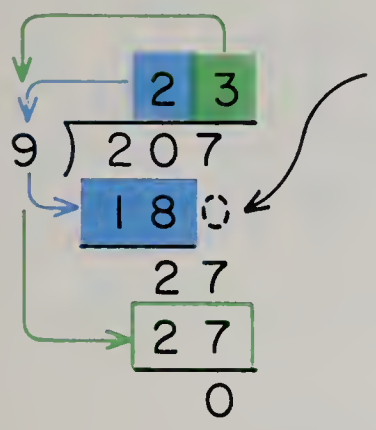
	Known Factor	Product	Multiples of Known Factor			Information about Missing Factor (If any)			
			x 10	x 100	x 1000	Number of digits	Digit on left	Next digit on right	Next digit to right
1.	8	768	80	800	8,000	2	9	6	none
2.	3	2,865	30	300	3,000	3	9		
3.	5	1,400							
4.	9	783							
5.	3	783							
6.	11	396							
7.	15	3,225							
8.	45	2,025							
9.	7	5,439							
10.	27	783							
11.	2	5,318							
12.	19	11,533							
13.	125	1,125							
14.	36	16,416							
15.	89	8,722							
16.	48	27,648							
17.	55	3,025							
18.	91	7,826							
19.	52	7,696							
20.	72	47,952							
21.	11	7,194							
22.	56	27,944							

is reserved for multiples of 10; is reserved for numbers less than 10.

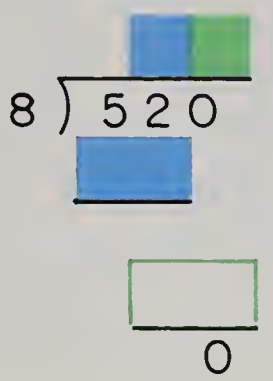
We agree that any number in the light green space is to be multiplied by 100.

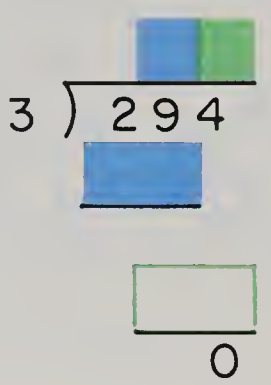
And that any number written in the blue space is to be multiplied by 10.

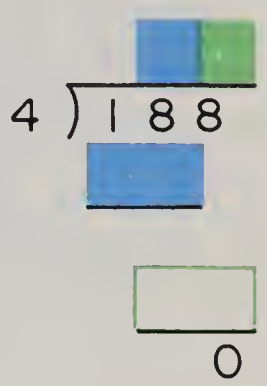
Dark green is still reserved for numbers less than 10.

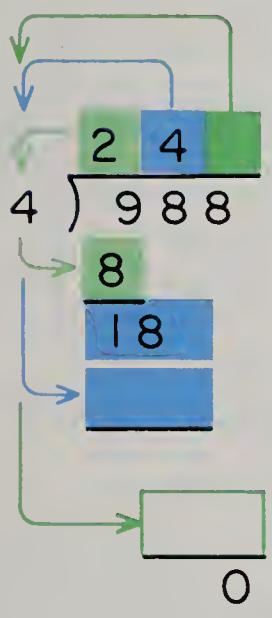


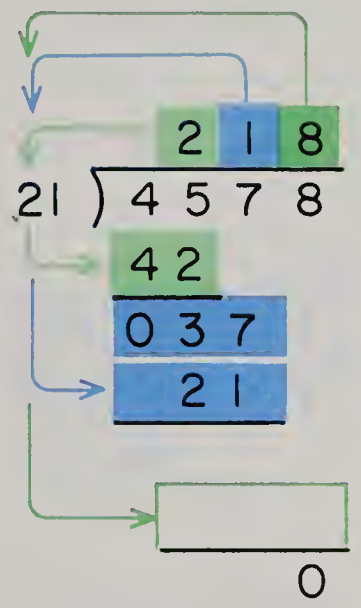
Write this as long as it helps you.

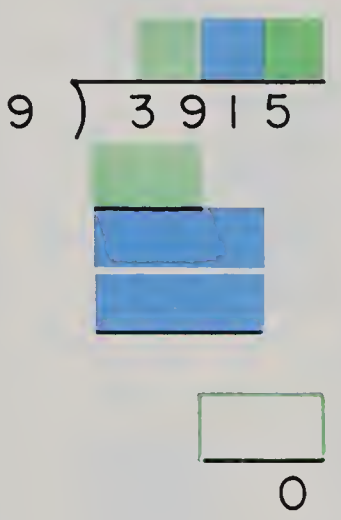


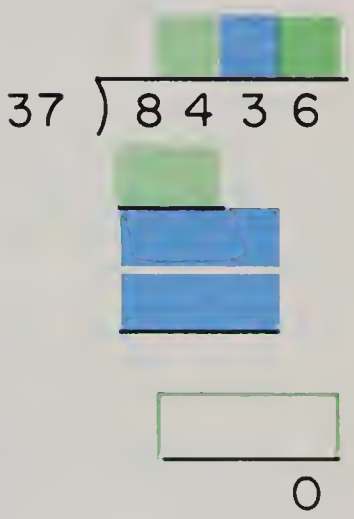


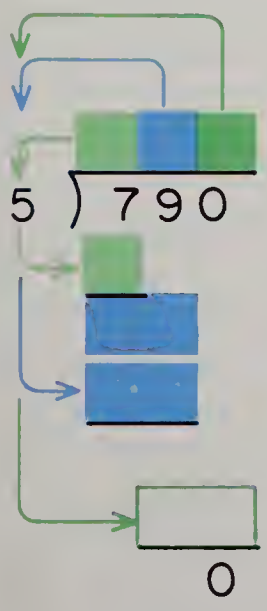


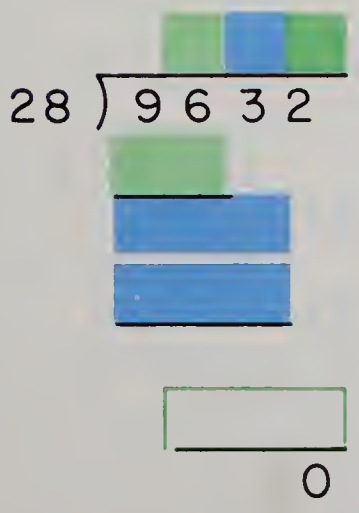


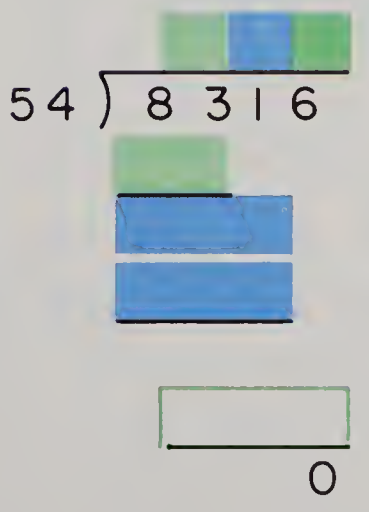


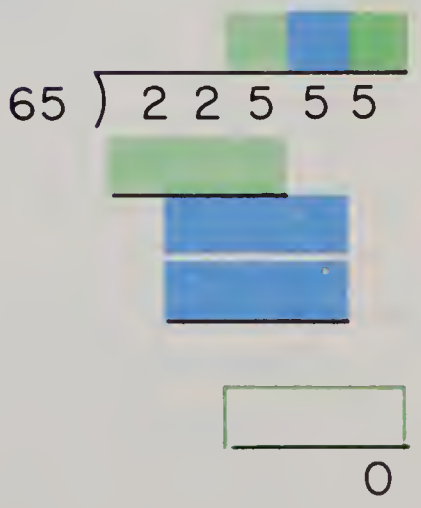


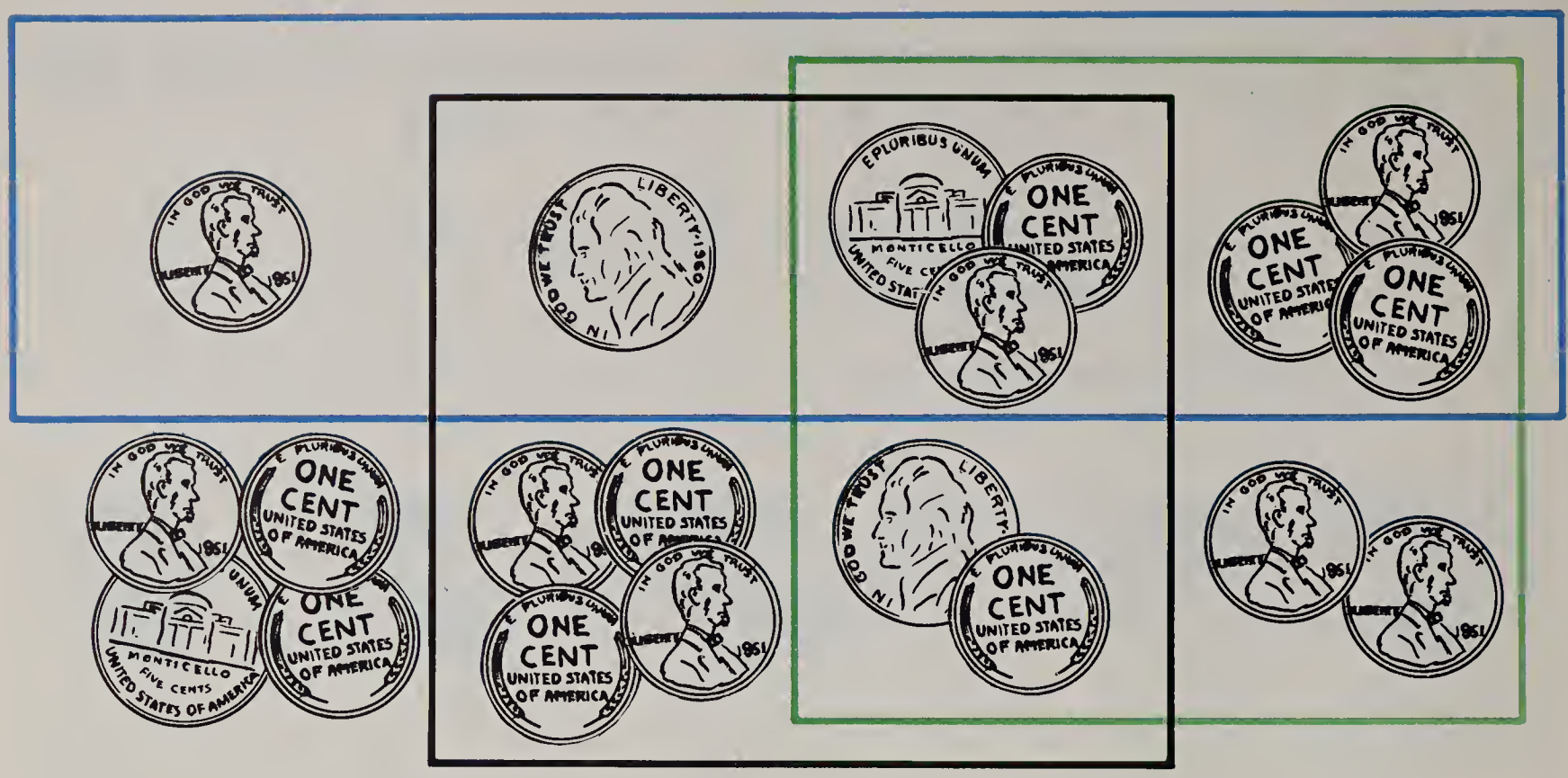










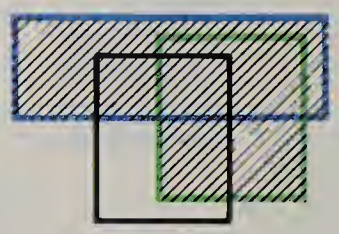


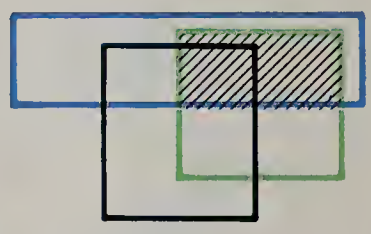
Let us agree that —

"In Green AND Blue" means "In both"
"In Green OR Blue" means "In either"

	Grey	Green	Blue	Value of Coins
Only in Blue, not in Green OR Grey				¢
Only in Green				¢
In Blue AND Green, not Grey				¢
Only in Grey				¢
Grey AND Blue, but not Green				¢
In Grey AND Green, but not Blue				¢
In Grey AND Green, AND Blue				¢
Not in any				¢

In Green AND Blue →





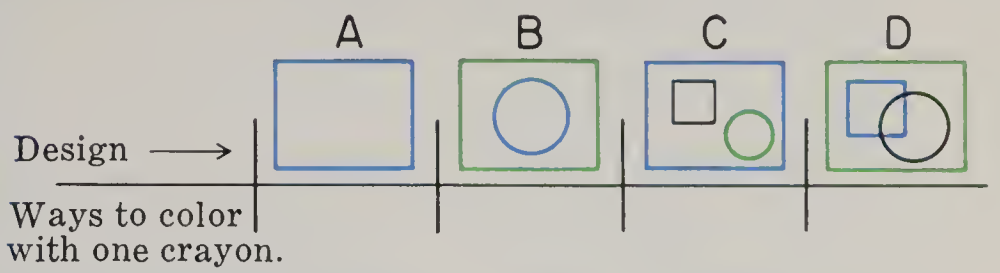
← In Green OR Blue

	Value of Coins
Green AND Blue	¢
Green OR Blue	¢
Green AND Grey	¢
Green OR Grey	¢
Blue AND Grey	¢
Blue OR Grey	¢
Blue AND Grey AND Green	¢
Blue OR Grey OR Green	¢
Grey, but not in Green AND Blue	¢

	Value of Coins
Not in any OR only in Blue	¢
Only in Grey OR only in Green	¢
Only in Grey AND only in Green	¢

How many different ways to color or not color each design — using only one crayon?

A

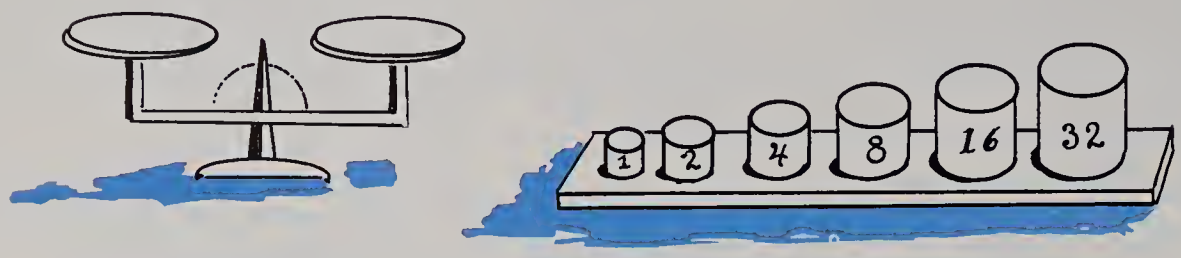


B

C

D

Jim worked in a chemical laboratory. One of his jobs was to package various chemicals. Orders always came to him for ounces of this or that.



He had only the weights shown — 1, 2, 4, 8, 16, and 32 ounces.

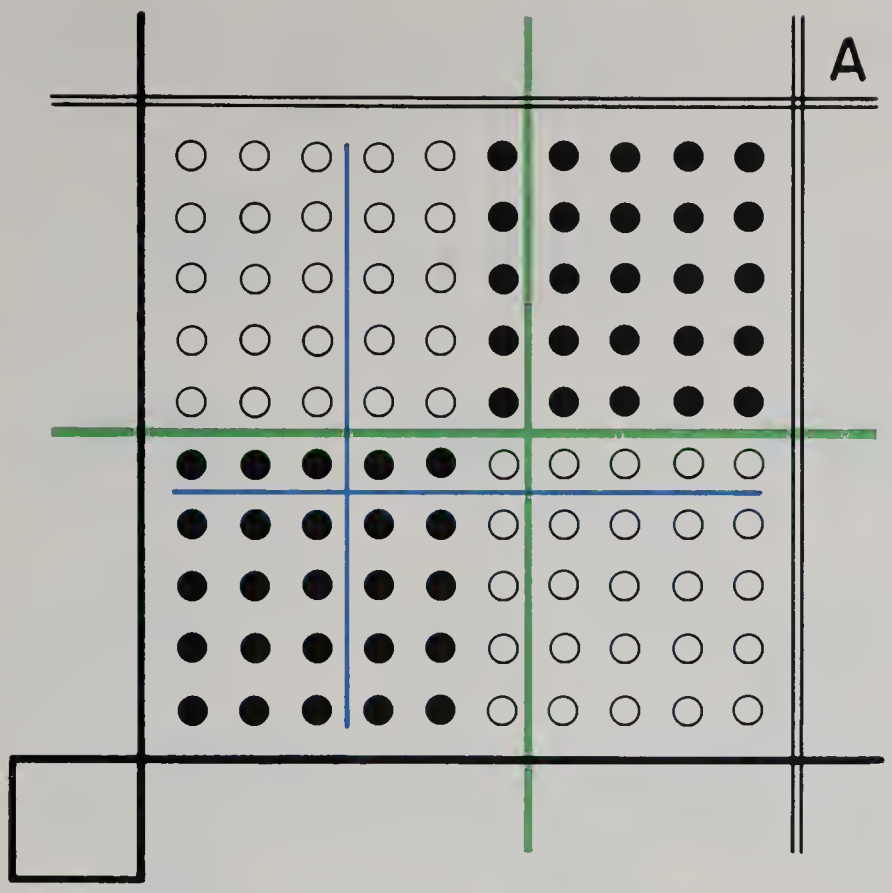
He knew that there are 16 ounces in one pound.

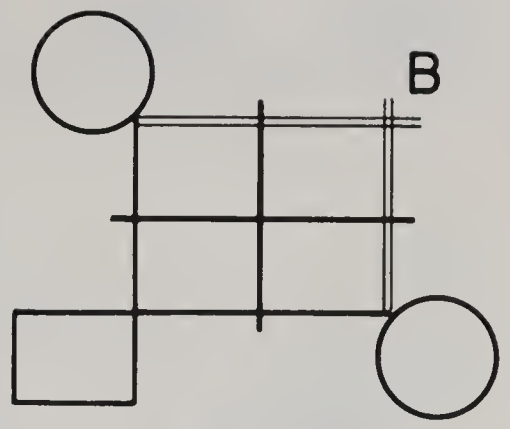
Some orders he received are listed below. What weights would he use for each? How many of the amounts could he weigh out with the six weights he had? Check the weights used as in example A.

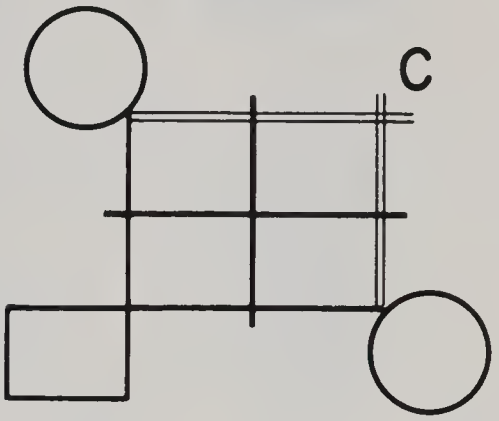
Orders	Weights — in ounces 32	16	8	4	2	1
A. 6 ounces of sodium				✓	✓	
B. 3 ounces of copper						
C. 8 ounces of sulphur						
D. 7 ounces of silver						
E. 11 ounces of chlorine						
F. 1 pound of zinc						
G. $1\frac{1}{2}$ pounds of aluminum						
H. 15 ounces of mercury						
I. $\frac{1}{2}$ pound of carbon						
J. $\frac{3}{4}$ pound of iron						
K. 27 ounces of calcium						
L. 3 lb., 9 oz. of lead						

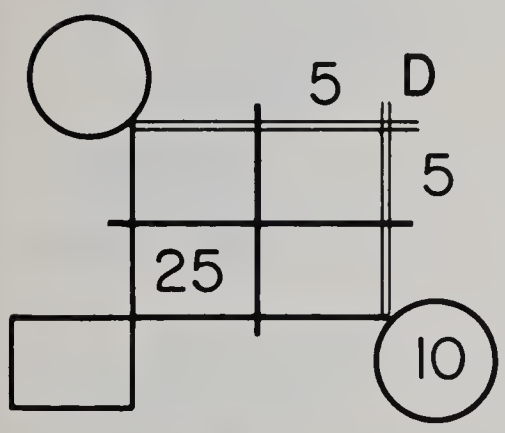
Jim made a chart so he would not make mistakes. It was like the one on the right. He put a big dot (●) to show the weights needed. Complete these charts.

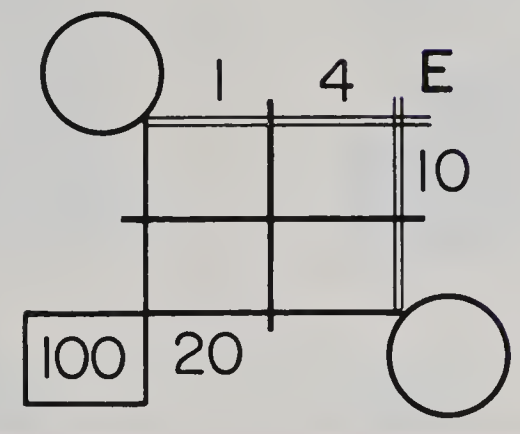
Weights in ounces 32	16	8	4	2	1	oz.
					●	1
				●		2
				●	●	3
						4
						5
			●	●		6
						7
						8
						9
						10
						11
						12
						13
						14
						15
						16
						17
						18
						19
						20
						21
						22
						23
						24
						25
						26
						27
						28
						29
						30
						31
						32
						33
						34
						35
						36
						37
						38
						39

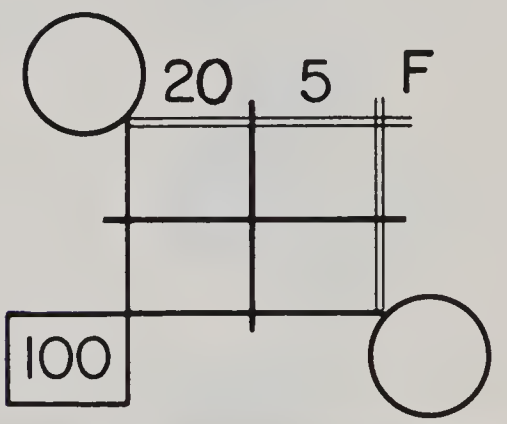

A

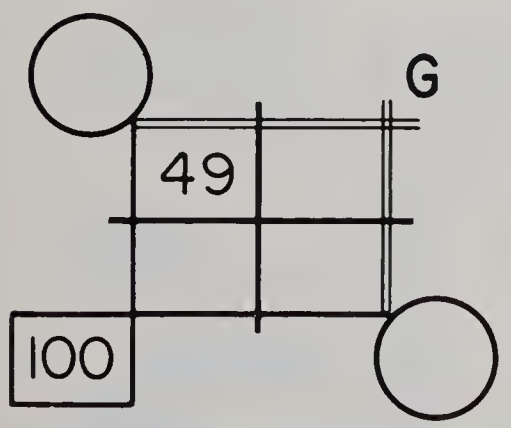

B

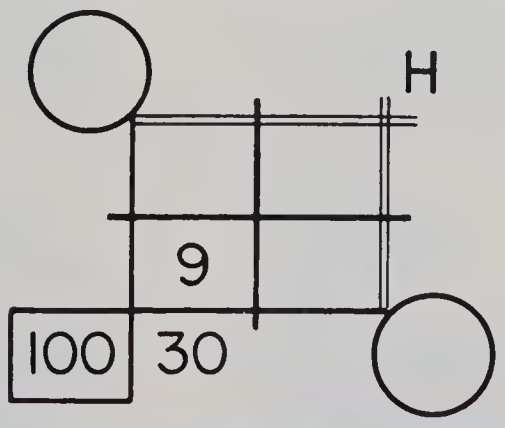

C

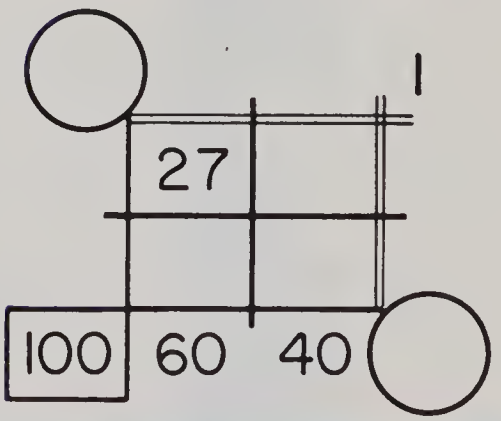

5
D
5
25
10

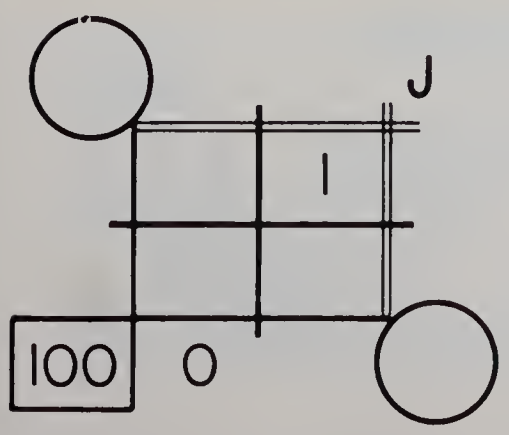

1
4
E
10
100
20

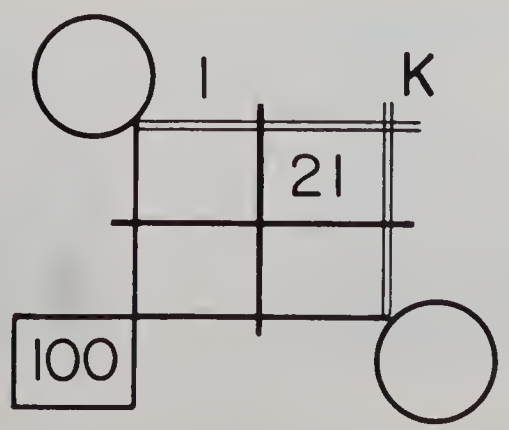

20
5
F
100

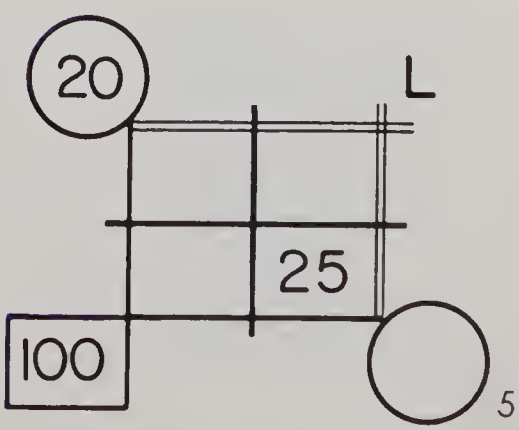

G
49
100


H
9
100
30


I
27
100
60
40


J
1
100
0

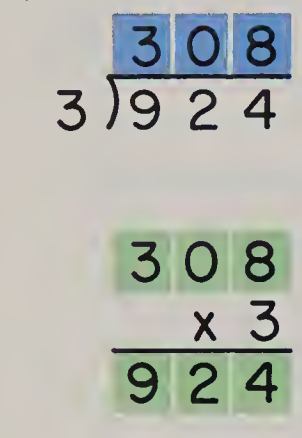

1
K
21
100


20
L
25
100

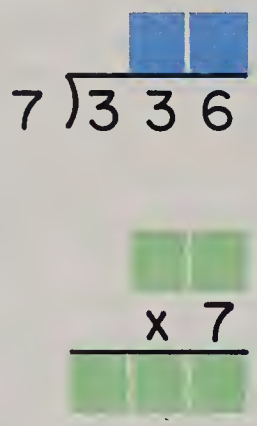
Another Method For Working Division Problems: A single digit is written in each place indicated by a blue block.

A means many people use to check their work is indicated by the green blocks.

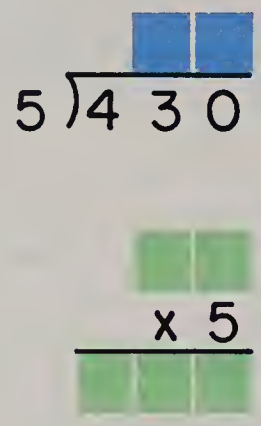
a.



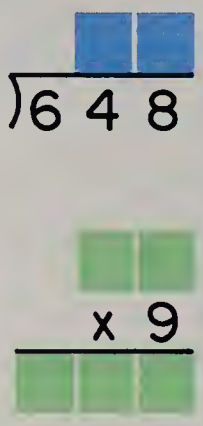
b.



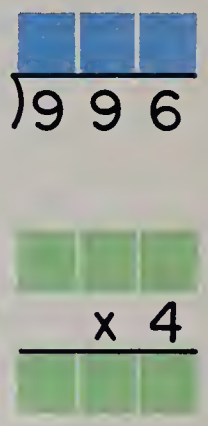
c.



d.

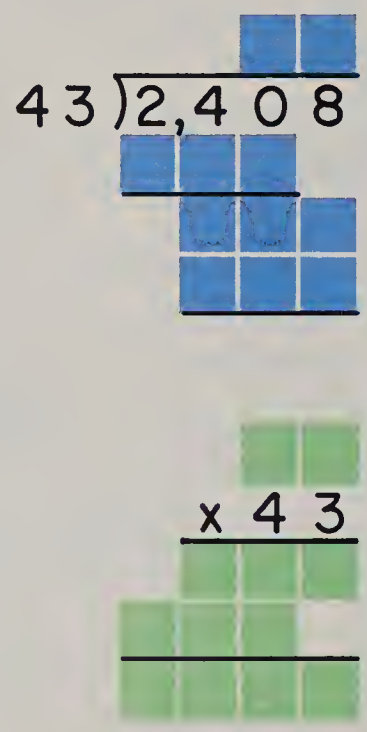


e.

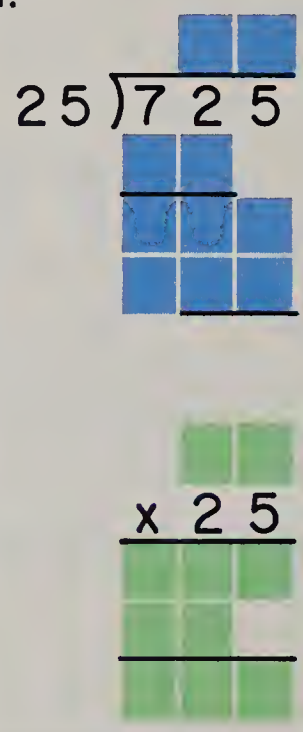


f.

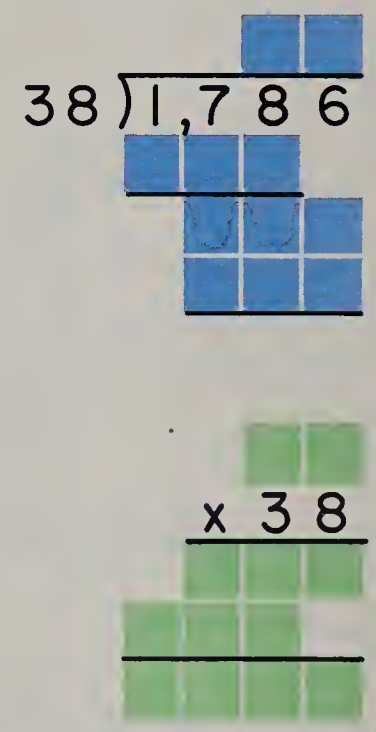
g.



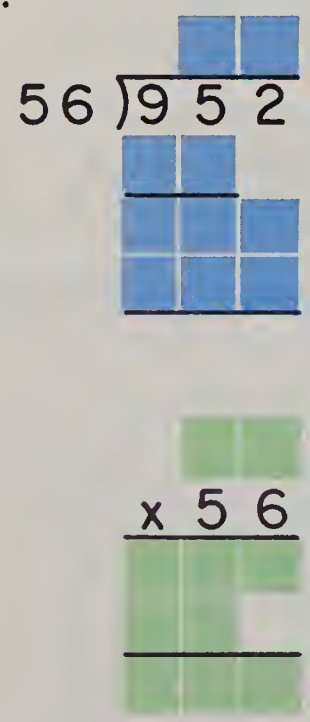
h.



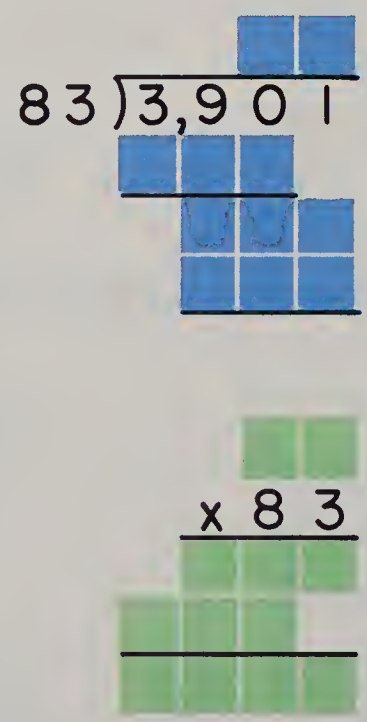
i.



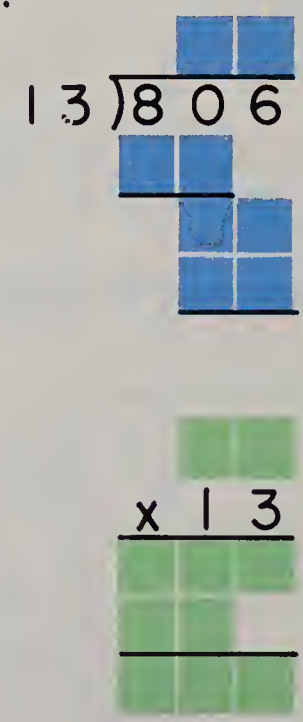
j.



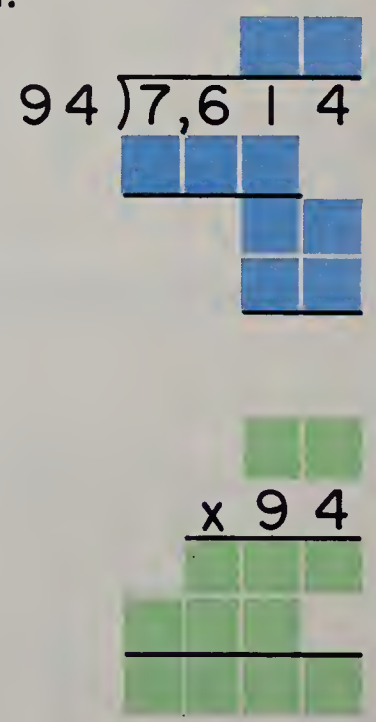
k.



l.

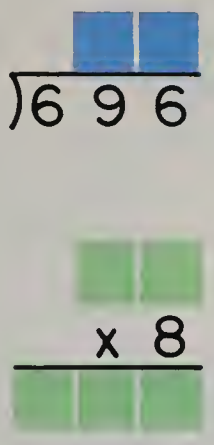


m.

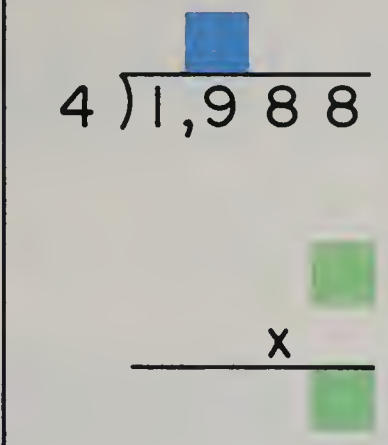


On this page, some of the blue and green spaces have been omitted.

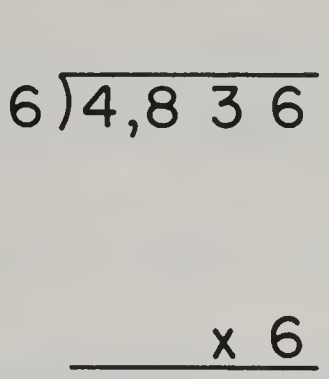
n.



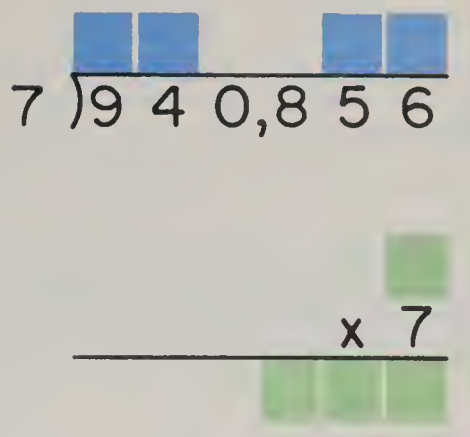
o.



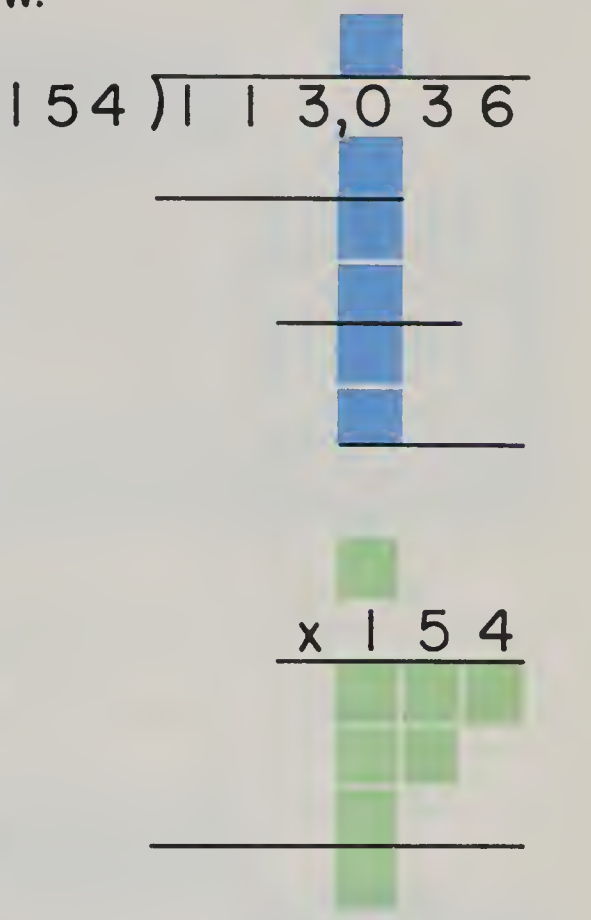
p.



v.

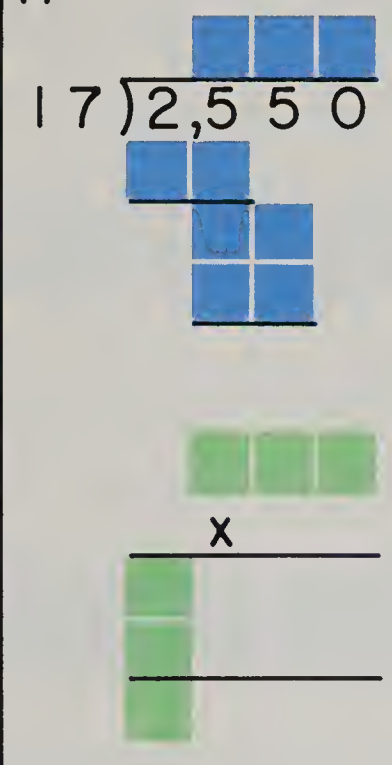


w.

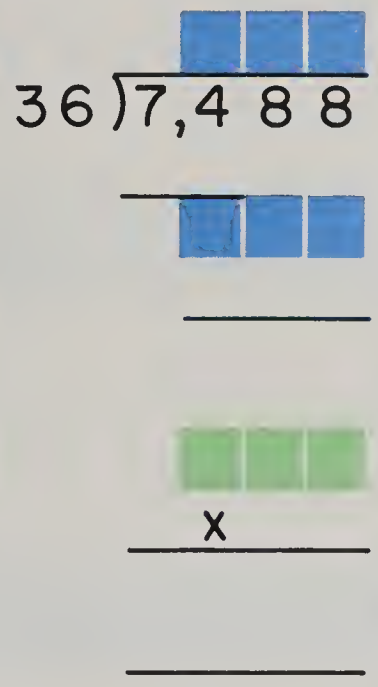


q.

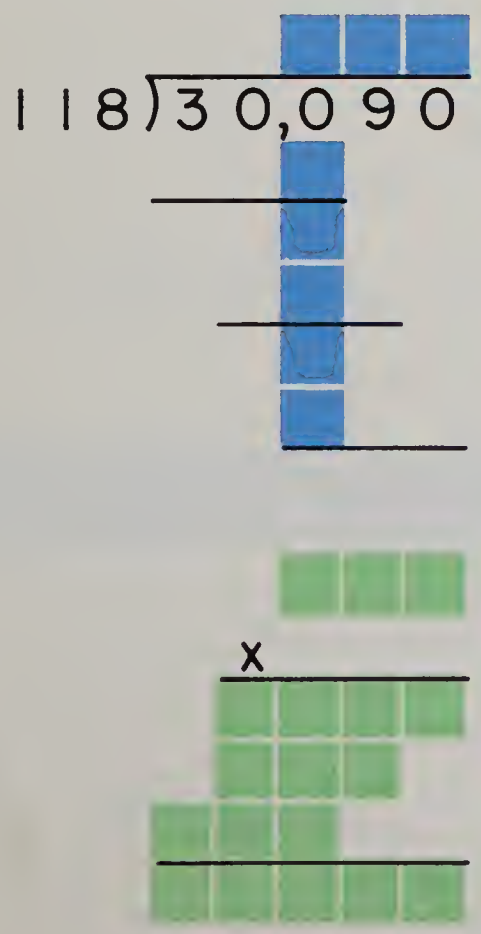
r.



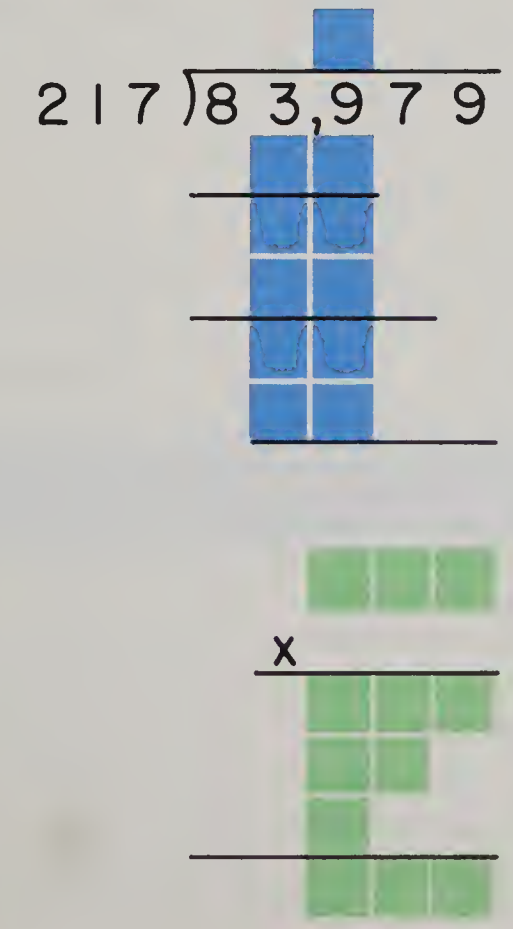
s.



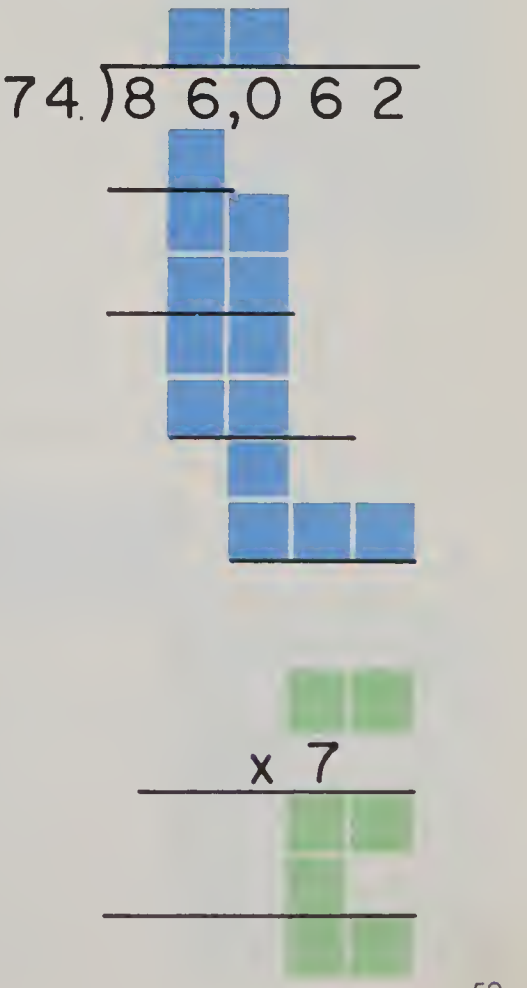
t.



u.



x.



How many minutes are there in half an hour? How many minutes in a quarter of an hour? How many in one-third of an hour?

How many minutes in a half and a quarter of an hour together?

How many minutes in a half of three hours? . . . In a third of three hours? . . . In a fourth of three hours? How many minutes in a half of three hours and a quarter of three hours together?

The chart below can be used to record answers to these and to many other questions.

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{2}+\frac{1}{4}$	$\frac{5}{6}$	$\frac{1}{3}+\frac{1}{2}$	$\frac{2}{3}+\frac{1}{6}$	$\frac{2}{3}-\frac{1}{6}$
An hour	30	20	15								
Three hours	90										
Four hours	120										
Eight hours											

How many cents in half of a dollar?

How many cents in a quarter of a dollar?

How many in one-tenth of a dollar?

How many cents in a half of five dollars?

	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{7}{10}$	$\frac{1}{2}+\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{5}+\frac{1}{10}$	$\frac{1}{2}-\frac{1}{5}$	$\frac{1}{4}-\frac{1}{5}$
One dollar	50	25									
Five dollars	250										
$1.60	80										
$3.40											

How many eggs in half of a dozen?

In one-fourth of a dozen?

How many in half of four dozen?

In half of five dozen?

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{1}{3}+\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{6}+\frac{1}{2}$	$\frac{7}{12}+\frac{1}{12}$	$\frac{3}{4}-\frac{1}{12}$
One dozen											
Seven dozen											
Six dozen											
Fourteen dozen											

How many is one-half of 24? How many is one-fourth of 24? . . . One-sixth of 24?

How many is one-half of 120? How many is two-thirds of 120?

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{3}{8}+\frac{1}{8}$	$\frac{1}{6}+\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{3}+\frac{1}{2}$	$\frac{1}{2}-\frac{1}{3}$
24	12	8	6							
120	60									
216	108									
168										
336										

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3}-\frac{1}{6}$	$\frac{1}{9}+\frac{2}{9}$	$\frac{1}{6}+\frac{1}{3}$	$\frac{1}{3}-\frac{1}{9}$	$\frac{1}{2}-\frac{1}{6}$	$\frac{5}{9}-\frac{2}{9}$
36	18	12								
180	90									
144	72									
252										
324										

Results in the charts on these two pages suggest that the following might be true statements.

$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ $\quad \frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ $\quad \frac{2}{3}+\frac{1}{6}=$ — $\quad \frac{2}{3}-\frac{1}{6}=$ —

$\frac{1}{2}+\frac{1}{5}=$ — $\quad \frac{1}{5}+\frac{1}{10}=$ — $\quad \frac{1}{2}-\frac{1}{5}=$ — $\quad \frac{1}{4}-\frac{1}{5}=$ —

$\frac{1}{3}+\frac{1}{4}=$ — $\quad \frac{1}{6}+\frac{1}{2}=$ — $\quad \frac{7}{12}+\frac{1}{12}=$ — $\quad \frac{3}{4}-\frac{1}{12}=$ —

$\frac{3}{8}+\frac{1}{8}=$ — $\quad \frac{1}{2}-\frac{1}{3}=$ — $\quad \frac{1}{6}+\frac{1}{3}=$ — $\quad \frac{5}{9}-\frac{2}{9}=$ —

Do you believe that these statements are true?

Mrs. Esser worked with Alec Marson's machine one night. Next day, she reported to the class.

"Without pushing the change button, I tried some examples at each of the eight dial settings. Each column below shows the results at one of the positions. I have labeled the columns so we can talk about them.

"I wrote a number in each corner of the green screen – and the machine immediately wrote a number in each corner of the blue screen. I think I know the rules it follows in each case. Can you discover the rules and complete the record?"

Rule R		Rule C		Rule B		Rule D	
I write	Machine writes						
7 3 / 5 1	10 5 / 6 3	7 3 / 5 1	8 1 / 8 1	7 3 / 5 1	12 3 / 4 1	7 3 / 5 1	4 1 / 4 1
10 5 / 30 15	15 1 / 45 3	10 5 / 30 15	25 5 / 35 7	10 5 / 30 15	40 2 / 20	10 5 / 30 15	5 1 / 15 3
12 8 / 4 2	20 10 / 6 3	12 8 / 4 2	14 / 12 6	12 8 / 4 2	16 / 10	12 8 / 4 2	4 2 / 2 1
6 3 / 18 9	9 1 / 27 3	6 3 / 18 9	15 / 7	6 3 / 18 9	12 /	6 3 / 18 9	1 / 3
20 12 / 9 7	2 / 1	20 12 / 9 7	9 / 21	20 12 / 9 7	29 / 19	20 12 / 9 7	/
40 10 / 35 25	5 / 6	40 10 / 35 25	/	40 10 / 35 25	/	40 10 / 35 25	/
30 24 / 60 12	3 / 4	30 24 / 60 12	/	30 24 / 60 12	/	30 24 / 60 12	/
45 15 / 36 30	/ 11	45 15 / 36 30	/	45 15 / 36 30	/	45 15 / 36 30	/
46 25 / 38 15	71 / 53	46 25 / 38 15	/	46 25 / 38 15	/	46 25 / 38 15	/

"Rule S and Rule A are tricky. More examples are given on the next page."

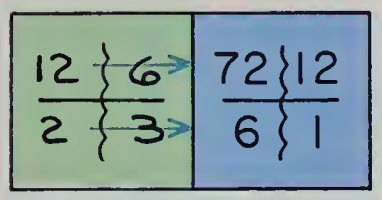
Rule P	Rule F	Rule S	Rule A
3 4 / 1 2 → 12 6 / 2 1	3 4 / 1 2 → 6 3 / 4 2	3 4 / 1 2 → 10 _ / 2 _	3 4 / 1 2 → 2 1 / 2 1
1 1 / 3 9 → 1 1 / 27 27	1 1 / 3 9 → 9 3 / 3 1	1 1 / 3 9 → 12 4 / 27 9	1 1 / 3 9 → 6 2 / 27 9
5 1 / 3 2 → 5 5 / 6 6	5 1 / 3 2 → 10 _ / 3 _	5 1 / 3 2 → 13 _ / 6 _	5 1 / 3 2 → 7 7 / 6 6
2 3 / 4 9 → _ _ / 36 6	2 3 / 4 9 → 18 _ / 12 2	2 3 / 4 9 → 30 _ / 36 _	2 3 / 4 9 → 6 _ / _ _
4 1 / 1 2 → _ _ / _ 1	4 1 / 1 2 → _ _ / 1 1	4 1 / 1 2 → 9 _ / _ _	4 1 / 1 2 → 7 _ / 2 _
5 2 / 1 1 → _ _ / _ _	5 2 / 1 1 → 5 _ / _ _	5 2 / 1 1 → 7 _ / 1 _	5 2 / 1 1 → 3 _ / 1 _
12 1 / 3 1 → _ 4 / _ _	12 1 / 3 1 → _ _ / _ _	12 1 / 3 1 → 15 _ / _ _	12 1 / 3 1 → 9 _ / 3 1
8 6 / 2 3 → _ _ / _ _	8 6 / 2 3 → _ _ / _ _	8 6 / 2 3 → 36 _ / _ _	8 6 / 2 3 → 12 _ / _ _
3 4 / 2 5 → _ _ / _ _	3 4 / 2 5 → _ _ / _ _	3 4 / 2 5 → _ _ / _ _	3 4 / 2 5 → 7 _ / _ _
4 2 / 12 6 → _ _ / _ _	4 2 / 12 6 → _ _ / _ _	4 2 / 12 6 → _ _ / _ _	4 2 / 12 6 → 0 _ / _ 72
8 9 / 3 6 → _ _ / _ _	8 9 / 3 6 → _ _ / _ _	8 9 / 3 6 → _ _ / _ _	8 9 / 3 6 → _ _ / _ _

The next day, Bill Martin said he had noticed something interesting about Rules P, F, S, and A.

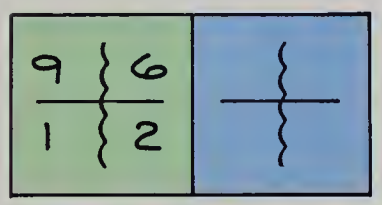
"In each column in each example, I tried pairs of numbers so that the number written on top divided by the number on the bottom is a whole number. Here are some examples:

$12 \div 2 = 6$ and $6 \div 3 = 2$."

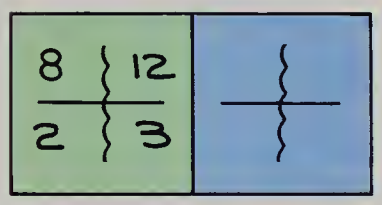
Rule P



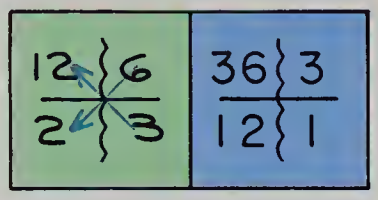
6 2 12 12



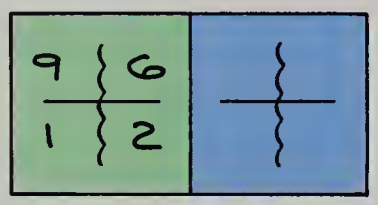
9 3

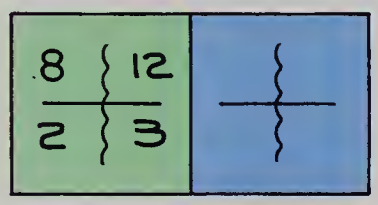


Rule F

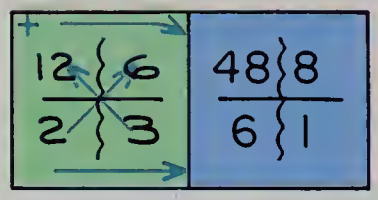


6 2 3 3

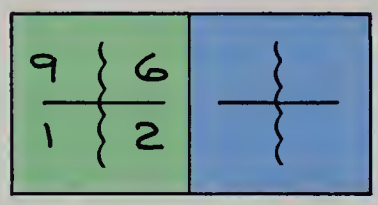


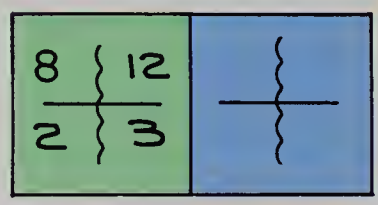


Rule S

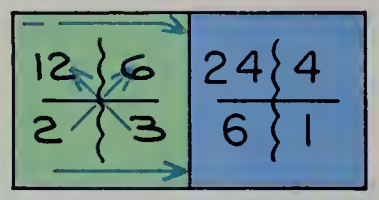


6 2 8 8

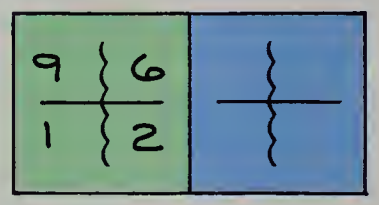


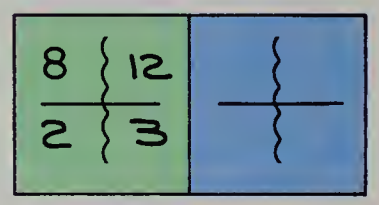


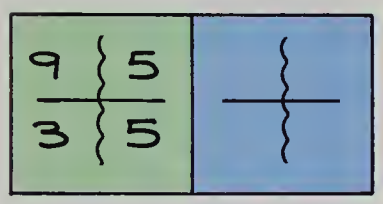
Rule A

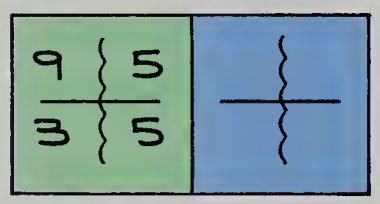


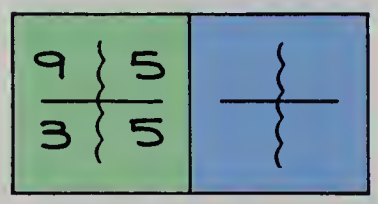
6 2 4 4

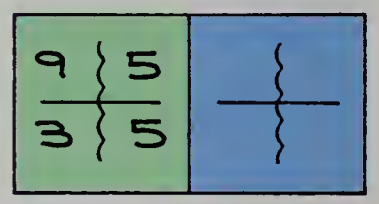


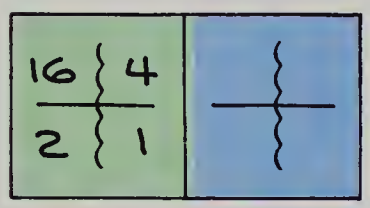


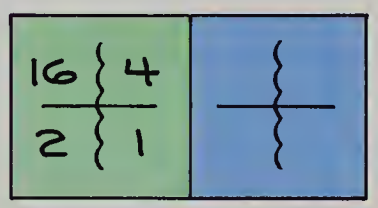


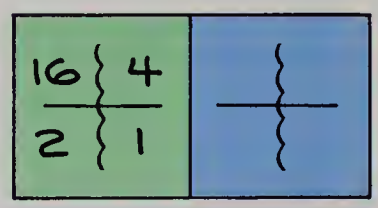


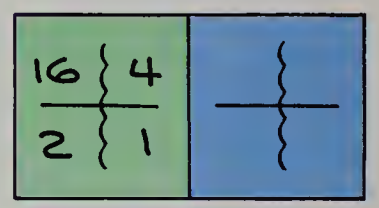












"When I arranged the results as below, I noticed an interesting fact."

$6 \times 2 = 12$	6 2 3	6 2 8	6 2 4
$9 \times 3 = 27$			
$4 \times 4 = 16$			
$3 \times 1 = 3$			
$8 \times 4 = 32$			

Jean Carlson then told of her own experiments.

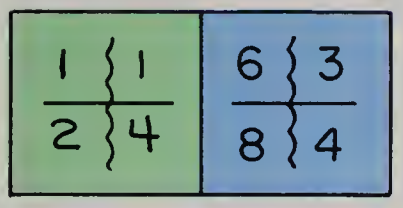
"I went back to our earlier work in which we found some results that seemed reasonable (see pages 54 and 55).

I tried Rules S and A as Alec's machine seems to operate.

"As you might expect, the machine would give exactly the same results as we got by other means.

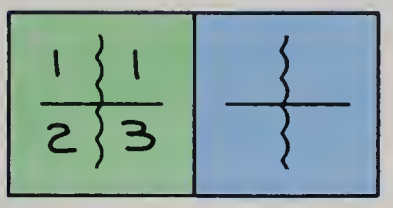
"Try them yourselves."

Rule S



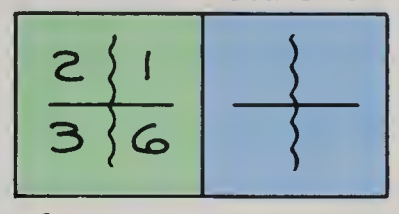
$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Rule S



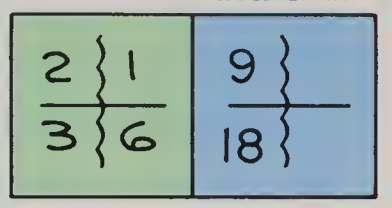
$\frac{1}{2} + \frac{1}{3} = —$

Rule S



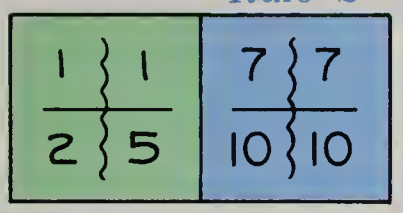
$\frac{2}{3} + \frac{1}{6} = —$

Rule A



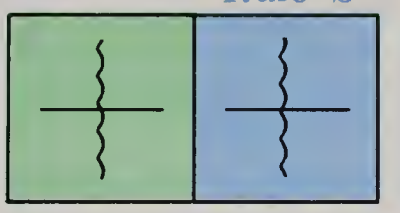
$\frac{2}{3} - \frac{1}{6} = —$

Rule S



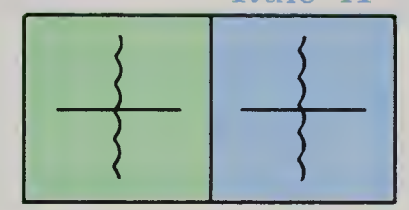
$\frac{1}{2} + \frac{1}{5} = —$

Rule S



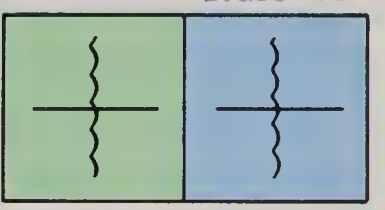
$\frac{1}{5} + \frac{1}{10} = —$

Rule A



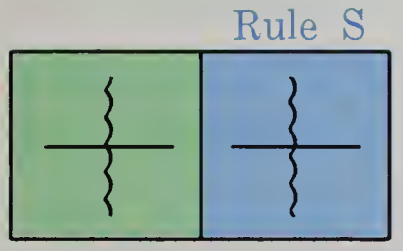
$\frac{1}{2} - \frac{1}{5} = —$

Rule A



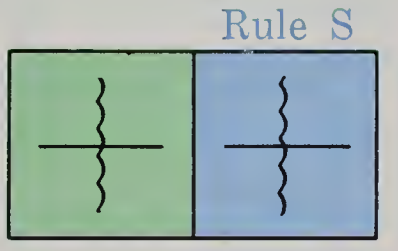
$\frac{1}{4} - \frac{1}{5} = —$

Rule S



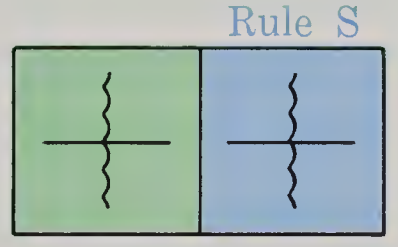
$\frac{1}{3} + \frac{1}{4} = —$

Rule S



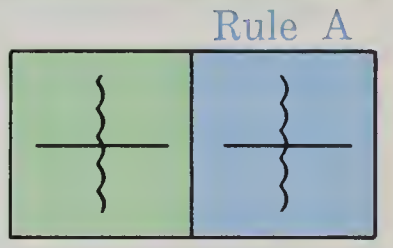
$\frac{1}{6} + \frac{1}{2} = —$

Rule S



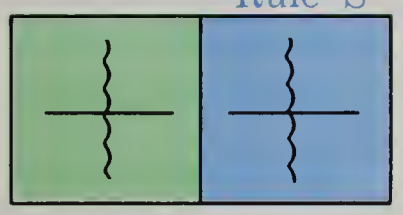
$\frac{7}{12} + \frac{1}{12} = —$

Rule A



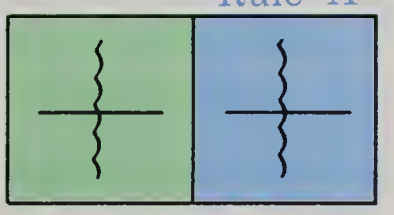
$\frac{3}{4} - \frac{1}{12} = —$

Rule S



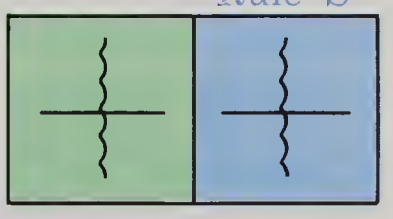
$\frac{3}{8} + \frac{1}{8} = —$

Rule A



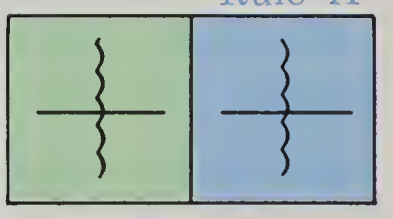
$\frac{1}{2} - \frac{1}{3} = —$

Rule S



$\frac{1}{6} + \frac{1}{2} = —$

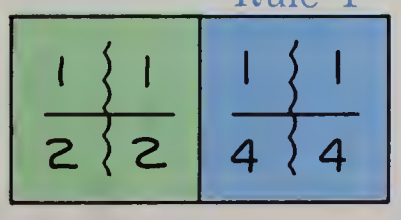
Rule A



$\frac{5}{9} - \frac{3}{9} = —$

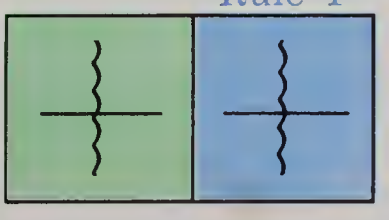
"Next," Jean continued, "I tried Rule P and Rule F in the same way. The results are not so obvious."

Rule P



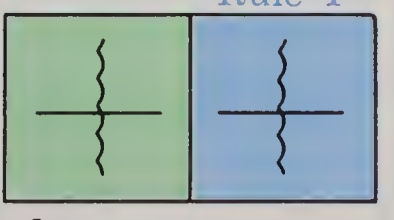
$\frac{1}{2} \times \frac{1}{2} = —$

Rule F



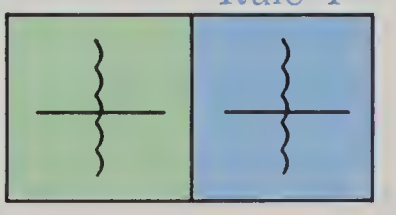
$\frac{1}{2} \div \frac{1}{2} =$

Rule P



$\frac{2}{1} \times \frac{1}{3} = —$

Rule F



$\frac{2}{1} \div \frac{1}{3} =$

Let's Talk About Dollars and Pies.	Patterns suggested by the numbers used:
Half of two dollars is _____ dollar.	a. $\frac{1}{2}$, $\frac{2}{\quad}$ → $\frac{1}{\quad}$
Half of one dollar is _____ of a dollar.	b. $\frac{1}{2}$, $\frac{1}{\quad}$ → $\frac{\quad}{\quad}$
Half of one-half dollar is _____ of a dollar.	c. $\frac{1}{2}$, $\frac{1}{2}$ → $\frac{\quad}{\quad}$
One-fourth of eight dollars is _____ dollars.	d. $\frac{1}{4}$, $\frac{8}{\quad}$ → $\frac{\quad}{\quad}$
One-fourth of two dollars is _____ of a dollar.	e. $\frac{1}{4}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
One-fourth of one pie is _____ of a pie.	f. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
One-fourth of one-half pie is _____ of a pie.	g. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
One-third of a half of a pie is _____ of a pie.	h. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$

Let's Talk About Half-dozens.	Patterns suggested:
Four of them (half-dozens) is _____ dozen.	i. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
Two of them (half-dozens) is _____ dozen.	j. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
One-third of a half-dozen is _____ of a dozen.	k. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$
Two-thirds of a half-dozen is _____ of a dozen.	l. $\frac{\quad}{\quad}$, $\frac{\quad}{\quad}$ → $\frac{\quad}{\quad}$

In the examples above, it seems that "," and "of" and "times" and "×" are practically interchangeable.

$\frac{1}{2}$, $\frac{2}{1}$ → $\frac{\quad}{\quad}$ and $\frac{1}{2}$ of $\frac{1}{2}$ → $\frac{\quad}{\quad}$ and $\frac{1}{2}$ times $\frac{1}{1}$ → $\frac{\quad}{\quad}$ and $\frac{1}{4} \times \frac{8}{1}$ → $\frac{\quad}{\quad}$

The last example could easily be changed to use only the well known symbols of arithmetic: $\frac{1}{4} \times \frac{8}{1} = \frac{2}{1}$. Complete the following:

$\frac{1}{4} \times \frac{1}{2} = \frac{\quad}{\quad}$ $\quad$ $\frac{2}{3} \times \frac{1}{1} = \frac{\quad}{\quad}$ $\quad$ $\frac{\quad}{\quad} \times \frac{1}{2} = \frac{1}{2}$ $\quad$ $\frac{\quad}{\quad} \times \frac{1}{2} = \frac{1}{4}$ $\quad$ $\frac{\quad}{\quad} \times \frac{1}{4} = \frac{1}{2}$

$\frac{1}{3} \times \frac{1}{2} = \frac{\quad}{\quad}$ $\quad$ $\frac{1}{2} \times \frac{2}{3} = \frac{\quad}{\quad}$ $\quad$ $\frac{4}{1} \times \frac{\quad}{\quad} = \frac{2}{1}$ $\quad$ $\frac{8}{1} \times \frac{\quad}{\quad} = \frac{2}{1}$ $\quad$ $\frac{1}{4} \times \frac{\quad}{\quad} = \frac{1}{8}$

Let's Talk About Dollars and Pies.	Patterns suggested:
In one dollar, how many half-dollars are there? Ans. _____	A. $\frac{1}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$
In one-half dollar, how many dollars are there? Ans. _____	B. $\frac{1}{2}, \frac{1}{1} \rightarrow$ —
In a quarter, how many half-dollars are there? Ans. _____	C. $\frac{1}{4}, \frac{1}{2} \rightarrow$ —
In two dollars, how many quarters are there? Ans. _____	D. $\frac{2}{1}$, — → —
In a half-dollar, how many quarters are there? Ans. _____	E. —, — → —
In a fourth of a pie, how many fourths are there? Ans. _____	F. —, — → —
In one-eighth of a pie, how many fourths are there? Ans. _____	G. —, — → —
In a sixth of a pie, how many thirds are there? Ans. _____	H. —, — → —

Let's Talk About Half-dozens.	Patterns suggested:
In two dozen, how many half-dozens are there? Ans. _____	I. $\frac{2}{1}$, — → —
In one dozen, how many half-dozens are there? Ans. _____	J. $\frac{1}{1}$, — → —
In one-sixth of a dozen, how many one-third dozens are there? Ans. _____	K. $\frac{1}{6}, \frac{1}{3} \rightarrow$ —
In one-third of a dozen, how many half-dozens are there? Ans. _____	L. $\frac{1}{3}$, — → —

In each pattern above, it seems that "," and "divided by" and "÷" are practically interchangeable.

$\frac{1}{1}, \frac{1}{2} \rightarrow$ — and $\frac{2}{1}$ divided by $\frac{1}{4} \rightarrow \frac{8}{1}$ and $\frac{1}{4} \div \frac{1}{4} \rightarrow \frac{1}{1}$ and $\frac{1}{2} \div \frac{1}{4} \rightarrow \frac{2}{1}$

The last examples could easily be changed to use only well known symbols of arithmetic. Complete the following:

$\frac{2}{1} \div \frac{2}{1} =$ — $\frac{2}{1} \div \frac{1}{1} =$ — $\frac{2}{1} \div \frac{1}{2} =$ — $\frac{2}{1} \div \frac{1}{4} =$ — $\frac{3}{1} \div \frac{1}{4} =$ —

$\frac{1}{2} \div \frac{1}{1} =$ — $\frac{1}{2} \div \frac{1}{2} =$ — $\frac{1}{2} \div \frac{1}{4} =$ — $\frac{1}{4} \div$ — $= \frac{1}{1}$ $\frac{1}{2} \div$ — $= \frac{1}{2}$

Mathematical Headlines that follow a pattern:	Write sentences for which the headline on the left is appropriate.
$\frac{2}{1}, \frac{1}{2} \rightarrow \frac{1}{1}$	Two half-gallons is the same as . . .
$\frac{6}{1}, \frac{1}{3} \rightarrow \frac{2}{1}$	
$\frac{1}{2}, \frac{8}{1} \rightarrow \frac{4}{1}$	
$\frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{4}$	
$\frac{1}{4}, \frac{2}{1} \rightarrow \frac{1}{2}$	
$\frac{1}{2}, \frac{1}{4} \rightarrow \frac{1}{8}$	
$\frac{1}{2}, \frac{1}{3} \rightarrow \frac{1}{6}$	
$\frac{1}{3}, \frac{2}{1} \rightarrow$ —	
$\frac{1}{4}, \frac{12}{1} \rightarrow \frac{3}{1}$	
$\frac{3}{4}, \frac{8}{1} \rightarrow$ —	
$\frac{2}{3}, \frac{9}{1} \rightarrow$ —	
$\frac{5}{1}, \frac{1}{4} \rightarrow$ —	
$\frac{6}{1}, \frac{2}{3} \rightarrow$ —	

Sentences below should reflect the idea suggested by the headline on the right:	Mathematical Headlines that follow a pattern;
I had two dollars. I asked for change. The bank teller gave me all quarters, ____ of them.	$\frac{2}{1}, \frac{1}{4} \rightarrow$ —
I had half of a pie. I cut it into pieces that were $\frac{1}{4}$ of a pie. Then I had ____ pieces.	$\frac{1}{2}, \frac{1}{4} \rightarrow$ —
June had a piece of ribbon 12" long. She cut it into 4 pieces. Each piece was ____ long.	$\frac{12}{1}, \frac{4}{1} \rightarrow$ —
Mrs. Jones bought two chocolate bars for the party. Each of the six guests got ____ of a candy bar.	$\frac{2}{1}, \frac{6}{1} \rightarrow$ —
There were 3 pies. They were cut so that each piece was one-sixth of a pie. I counted the pieces. There were ___.	$\frac{3}{1}, \frac{1}{6} \rightarrow$ —
A group of girls earned two dollars. The four girls shared it evenly. Each girl _____	$\frac{2}{1}, \frac{4}{1} \rightarrow$ —
I have an eighth of a gallon. How many half-gallons is that? It is _____ of a half-gallon.	$\frac{1}{8}, \frac{1}{2} \rightarrow$ —
Mary worked for 8 hours. It took two hours to make each apron.	$\frac{8}{1}, \frac{2}{1} \rightarrow 4$
Betty worked 8 hours. In each half-hour, she made ____ ____	$\frac{8}{1}, \frac{1}{2} \rightarrow$ —
In two miles, how many half-miles are there? There are ____	$\frac{2}{1}, \frac{1}{2} \rightarrow$ —
In one yard, ____ ____ ____ ____ ____ ____ ____ ____	$\frac{1}{1}, \frac{1}{2} \rightarrow$ —
In a half-dozen, ____ ____ ____ ____ ____ ____ ____ ____	$\frac{1}{2}, \frac{1}{2} \rightarrow$ —
In a quarter of an hour, ____ ____ ____ ____ ____ ____	$\frac{1}{4}, \frac{1}{2} \rightarrow$ —

Insert +, −, X or ÷ to make the following into true statements.

$\frac{9}{3}\ \frac{4}{2} = \frac{5}{1}$	$\frac{1}{1} = \frac{3}{4}\ \frac{1}{4}$	$\frac{2}{3}\ \frac{3}{4} = \frac{1}{2}$	$\frac{3}{1} = \frac{2}{1}\ \frac{2}{3}$
$\frac{10}{5}\ \frac{4}{2} = \frac{1}{1}$	$\frac{1}{6} = \frac{3}{6}\ \frac{1}{3}$	$\frac{1}{4} = \frac{3}{8}\ \frac{1}{8}$	$\frac{3}{8}\ \frac{5}{8} = \frac{1}{1}$
$\frac{12}{3}\ \frac{4}{4} = \frac{4}{1}$	$\frac{1}{2} = \frac{1}{4}\ \frac{1}{4}$	$\frac{3}{1}\ \frac{1}{2} = \frac{6}{1}$	$\frac{5}{4}\ \frac{2}{3} = \frac{5}{6}$
$\frac{7}{1}\ \frac{6}{3} = \frac{5}{1}$	$\frac{3}{1} = \frac{3}{8}\ \frac{1}{8}$	$\frac{1}{2} = \frac{7}{4}\ \frac{5}{4}$	$\frac{1}{8} = \frac{7}{8}\ \frac{3}{4}$

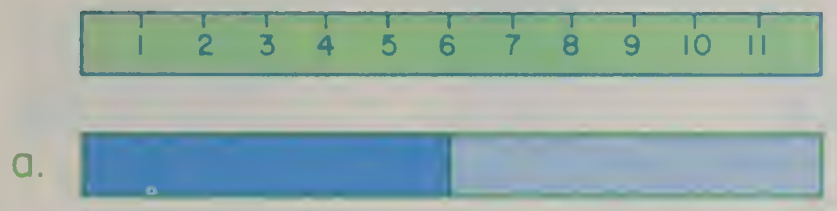
Insert two of these signs: +, −, X, ÷ or = to make each of the following into a true statement:

$\frac{14}{2} + \frac{9}{3} = \frac{10}{1}$	$\frac{7}{8}\ \frac{1}{8}\ \frac{1}{1}$	$\frac{1}{2}\ \frac{1}{8}\ \frac{3}{8}$	$\frac{4}{6}\ \frac{1}{3}\ \frac{1}{1}$
$\frac{4}{2}\ \frac{3}{1}\ \frac{6}{1}$	$\frac{18}{3}\ \frac{8}{4}\ \frac{3}{1}$	$\frac{2}{3}\ \frac{2}{6}\ \frac{2}{9}$	$\frac{4}{2}\ \frac{3}{6}\ \frac{1}{1}$
$\frac{3}{4}\ \frac{1}{1}\ \frac{1}{4}$	$\frac{1}{4}\ \frac{3}{8}\ \frac{1}{8}$	$\frac{9}{8}\ \frac{3}{8}\ \frac{3}{4}$	$\frac{0}{7}\ \frac{3}{9}\ \frac{1}{3}$
$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{4}$	$\frac{16}{1}\ \frac{8}{1}\ \frac{1}{2}$	$\frac{2}{3}\ \frac{2}{5}\ \frac{3}{5}$	$\frac{7}{8}\ \frac{2}{3}\ \frac{21}{16}$

Make the following true by using needed signs: +, −, X, ÷ or =

$\frac{6}{2} \times \frac{9}{3} = \frac{54}{6} = \frac{9}{1}$	$\frac{8}{2}\ \frac{3}{1} = \frac{14}{2} = \frac{7}{1}$	$\frac{8}{2}\ \frac{3}{1}\ \frac{2}{2} = \frac{1}{1}$
$\frac{2}{1}\ \frac{1}{2} = \frac{4}{1} = \frac{4}{1}$	$\frac{9}{8}\ \frac{1}{8}\ \frac{64}{64}\ \frac{1}{1}$	$\frac{18}{3}\ \frac{4}{2} = \frac{36}{12}\ \frac{3}{1}$
$\frac{3}{4}\ \frac{1}{4}\ \frac{16}{16}\ \frac{1}{1}$	$\frac{7}{1}\ \frac{1}{3}\ \frac{21}{1}\ \frac{21}{1}$	$\frac{1}{8}\ \frac{1}{8}\ \frac{16}{64}\ \frac{1}{4}$
$\frac{1}{2}\ \frac{3}{16}\ \frac{10}{32}\ \frac{5}{16}$	$\frac{2}{3}\ \frac{1}{2}\ \frac{2}{6}\ \frac{1}{3}$	$\frac{2}{7}\ \frac{6}{3}\ \frac{12}{21}\ \frac{4}{7}$

Consider some of the many ways in which you could cut up a ruler into pieces that are the same length.



Different possibilities can be summarized in the chart below:

Ways of cutting a 12-inch ruler into pieces of the same length.

Length of pieces	6 Inches	4 Inches	12 Inches	3 Inches	2 Inches	1 Inch	$\frac{1}{2}$ Inch	$\frac{1}{4}$ Inch	$\frac{1}{8}$ Inch	$\frac{1}{3}$ Inch	$\frac{2}{3}$ Inch	$\frac{3}{4}$ Inch
Number of pieces												

Complete the following:

In twelve dollars there are _____ half-dollars and _____ quarters and _____ dimes and _____ nickels and _____ pennies.

If you can make a jump on the moon $\frac{1}{4}$ of a mile long, it would take _____ jumps to go 1 mile, _____ jumps to go a half-mile, _____ jump to go a quarter of a mile, _____ jumps to go two miles, 16 jumps to go _____ miles, and 12 jumps to go _____ miles.

If 12 gallons of milk is poured into half-gallon containers, it would fill _____ of them, or it would fill _____ one-gallon containers, or _____ quart containers, or _____ pint containers.

If a yardstick is cut up into pieces of equal length, there would be _____ pieces 3 inches long, _____ pieces $\frac{1}{2}$ inch long, or 18 pieces _____ long, and _____ pieces $\frac{1}{4}$ inch long.

Write examples to fit the following headlines:

$24 \div \frac{1}{2} =$

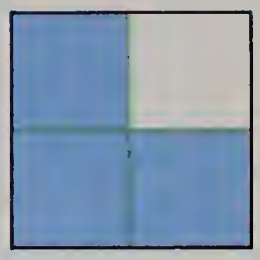
$6 = 2 \div$

$4 \div \frac{2}{3} =$

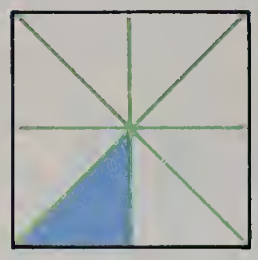
How many — of a tile are needed to cover $\frac{3}{4}$ of a tile?

$\frac{3}{4}$ of a tile

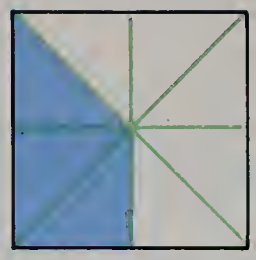
— of a tile



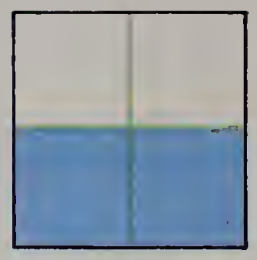
— of a tile



— of a tile



— of a tile



— of a tile

To cover $\frac{3}{4}$ of a piece of tile, it would take . . .

Number of pieces										
Size of each piece	$\frac{1}{4}$ of a tile	$\frac{3}{4}$ of a tile	$\frac{1}{8}$ of a tile	$\frac{3}{8}$ of a tile	$\frac{1}{2}$ of a tile	$\frac{1}{16}$ of a tile	$\frac{2}{16}$ of a tile	$\frac{4}{16}$ of a tile	$\frac{8}{16}$ of a tile	$\frac{12}{16}$ of a tile

This seems to indicate that . . .

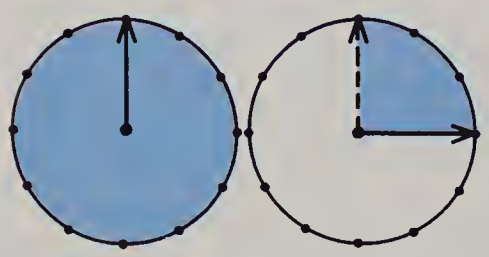
$\frac{3}{4} \div \frac{1}{4} = —$ $\frac{3}{4} \div \frac{1}{2} = —$ $\frac{3}{4} \div \frac{3}{4} = —$ $\frac{2}{4} \div \frac{1}{} = \frac{2}{1}$ $— \div \frac{1}{2} = \frac{3}{2}$

$\frac{6}{8} \div \frac{1}{8} = —$ $\frac{6}{8} \div \frac{2}{8} = —$ $\frac{6}{8} \div \frac{3}{8} = —$ $\frac{6}{8} \div \frac{4}{8} = —$ $\frac{6}{8} \div \frac{6}{8} = —$

A nice pattern pops out if we write these examples in a little different way:

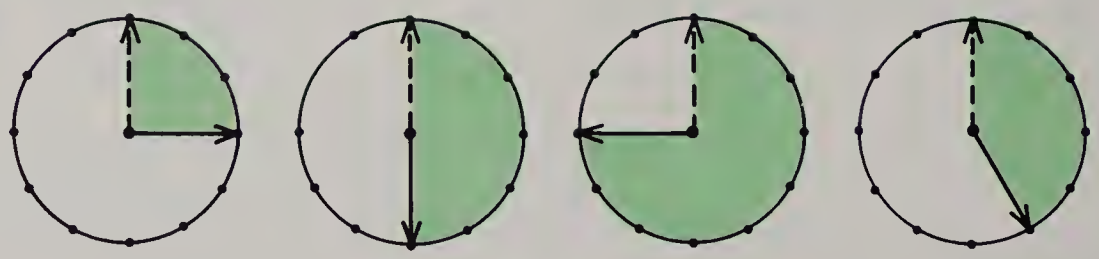
$\frac{3}{4} \div \frac{1}{4} = \frac{3}{1}$ $\frac{3}{4} \div \frac{}{4} = \frac{}{2}$ $\frac{3}{4} \div \frac{}{} = \frac{3}{3}$ $\frac{2}{4} \div \frac{1}{} = \frac{2}{1}$ $\frac{3}{4} \div \frac{2}{4} = \frac{}{2}$

$\frac{6}{8} \div \frac{1}{8} = \frac{}{1}$ $\frac{6}{8} \div \frac{2}{8} = \frac{}{2}$ $\frac{6}{8} \div \frac{3}{8} = \frac{}{3}$ $\frac{6}{8} \div \frac{4}{8} = \frac{}{4}$ $\frac{6}{8} \div \frac{6}{8} = \frac{}{6}$



$1\frac{1}{4}$ or $\frac{5}{4}$ hours

In $1\frac{1}{4}$ hours, how many ____ of an hour are there?



There are ___ quarter-hours in an hour and a quarter. In an hour and a quarter, there are _____ half-hours, there are ____ periods of 25 minutes, there are $1\frac{2}{3}$ periods of ___ of an hour. There are ____ twelfths of an hour in an hour and a quarter.

$\frac{5}{4} \div \frac{1}{4} = —$ $\frac{5}{4} \div \frac{1}{2} = —$ $\frac{5}{4} \div \frac{3}{4} = —$ $\frac{5}{4} \div \frac{5}{12} = —$ $\frac{5}{4} \div \frac{1}{12} = —$

$\frac{5}{4} \div \frac{1}{2} = \frac{5}{4} \div \frac{2}{4} = \frac{}{2}$ $\frac{5}{4} \div \frac{5}{12} = \frac{}{12} \div \frac{5}{12} = —$ $\frac{5}{4} \div \frac{1}{12} = \frac{}{12} \div \frac{}{12} = \frac{}{1}$

$\frac{5}{4} \div \frac{1}{8} = \frac{}{8} \div \frac{}{8} = \frac{}{1}$ $\frac{5}{4} \div \frac{1}{3} = \frac{}{12} \div \frac{}{12} = —$ $\frac{5}{4} \div \frac{1}{5} = — \div — = —$

Use =, > or < to make the following into true statements.

$\frac{1}{2} \div \frac{1}{4} \square \frac{2}{1}$	$\frac{3}{4} \div \frac{3}{8} \square \frac{3}{2}$	$\frac{5}{4} \div \frac{1}{2} \square \frac{5}{2}$	$\frac{7}{8} \div \frac{6}{8} \square \frac{1}{1}$	$\frac{1}{2} \div \frac{3}{8} \square \frac{1}{1}$
$\frac{15}{3} \div \frac{1}{5} \square \frac{5}{3}$	$\frac{5}{12} \div \frac{1}{2} \square \frac{1}{1}$	$\frac{7}{2} \div \frac{3}{2} \square \frac{3}{2}$	$\frac{3}{4} \div \frac{3}{8} \square \frac{2}{1}$	$\frac{9}{11} \div \frac{9}{11} \square \frac{5}{6}$
$\frac{9}{3} \div \frac{12}{2} \square \frac{1}{2}$	$\frac{1}{1} \div \frac{2}{5} \square \frac{7}{2}$	$\frac{5}{4} \div \frac{1}{3} \square \frac{10}{2}$	$\frac{2}{3} \div \frac{3}{4} \square \frac{1}{1}$	$\frac{6}{7} \div \frac{2}{7} \square \frac{3}{1}$
$\frac{5}{4} \div \frac{1}{3} \square \frac{4}{1}$	$\frac{1}{1} \div \frac{7}{8} \square \frac{1}{1}$	$\frac{2}{3} \div \frac{1}{1} \square \frac{7}{8}$	$\frac{1}{1} \div \frac{2}{3} \square \frac{7}{8}$	$\frac{7}{8} \div \frac{2}{3} \square \frac{1}{1}$

Complete the following headlines or stories, or both.

$\frac{7}{8} \div \frac{2}{3} \square 1$

"I have pails that hold $\frac{2}{3}$ of a gallon each. I want to know whether I can pour $\frac{7}{8}$ of a gallon into 1 pail. I know that I _____ because $\frac{7}{8}$ is _____ than $\frac{2}{3}$."

"There are 24 hours in a day, so $\frac{7}{8}$ of a day is ____ hours. Two-thirds of a day is _____ hours. There is certainly _____ (more or less) than one 16-hour period in 21 hours. In fact, there are 1 and — such periods."

$2 \div \frac{2}{3} = \square$

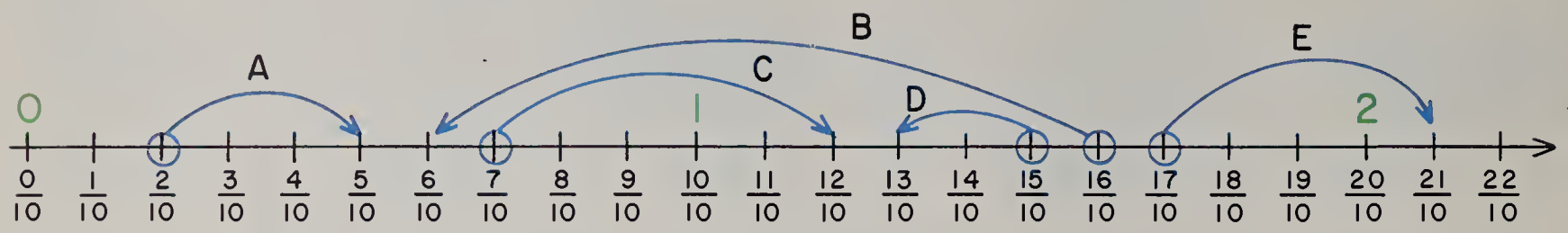
$2\frac{1}{8} \div \frac{1}{2} \square 5$

$3 \div 7 \square \frac{1}{2}$

$— \div \frac{1}{8} = \square 16$

$— \div \frac{1}{3} > \square 4\frac{1}{2}$

Perhaps, on some inhabited planet, children first learn of fractions on a number-line such as the one below. Jumps are shown to suggest addition and subtraction. Try some examples.



A $\frac{2}{10} + \frac{3}{10} = \frac{5}{10}$

B $\frac{16}{10} - \frac{10}{10} = \frac{\quad}{10}$

C $\frac{7}{10} + \frac{\quad}{10} = \frac{12}{10}$

D $\frac{15}{10} - \frac{\quad}{10} = \frac{13}{10}$

E $\frac{17}{10} + \frac{\quad}{10} = \frac{21}{10}$

F $\frac{1}{10} + \frac{\quad}{\quad} = \frac{10}{10}$

G $\frac{25}{10} - \frac{\quad}{10} = \frac{15}{10}$

H $\frac{100}{10} + \frac{25}{10} = \frac{\quad}{10}$

I $\frac{40}{10} - \frac{30}{10} = \frac{\quad}{10}$

You would not be surprised to learn that they developed a shorthand to save so much repetition of writing of 10 as denominator.

Can you discover their system?

a .2 + .3 = .5

b 1.6 − 1.0 = .6

c .7 + = 1.2

d 1.5 − = 1.3

e 1.7 + = 2.1

f .1 + = 1.0

g 2.5 − = 1.5

h 10.0 + 2.5 =

i 4.0 − 3.0 =

j 11.0 − 9.5 =

k 32.5 − 5.0 =

l .8 + = 3.0

m 10.2 + 3.9 =

n 8.4 + 8.4 =

o 10.3 − = 1.5

Of course, they could write whole numbers in this same way because:

1 = $\frac{10}{10}$ = 1.0 2 = $\frac{20}{10}$ = 2.0 3 = $\frac{\quad}{10}$ = 4 = $\frac{\quad}{10}$ =

7 = $\frac{\quad}{10}$ = 9 = $\frac{\quad}{10}$ = 10 = $\frac{\quad}{\quad}$ = 11 = $\frac{\quad}{\quad}$ =

J 2 + .3 = 2.3

K 2 + 1.3 = 3.3

L 2 + 2 = 4.0

M .3 + 1.4 =

N 1.6 + = 2.6

O 2 − .3 = 1.7

P 3.2 − 2 =

Q 5 − 3.4 =

R 2 − 1.9 =

S 8 − 3 =

T 5 + 10.2 =

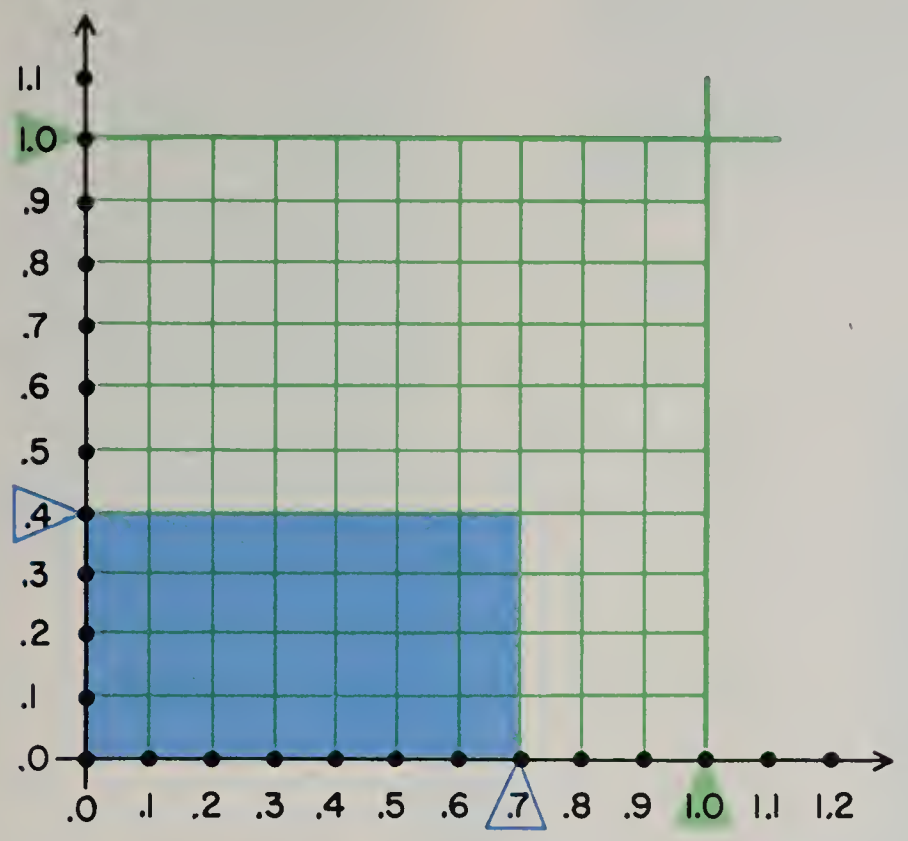
U 21.5 − 2 =

V 8.1 − .9 =

W .8 + 10 =

X 7.7 − 7 =

On this same planet, they would draw sketches to suggest the idea of multiplication and division. The diagrams might look like those below:



We know that $1 \times 1 = 1$

Also, we know that $\frac{10}{10} = 1.0 = 1$

And we know that $\frac{10}{10} \times \frac{10}{10} = \frac{100}{100} = 1$

The square made of heavy green and black lines is a diagram that suggests:

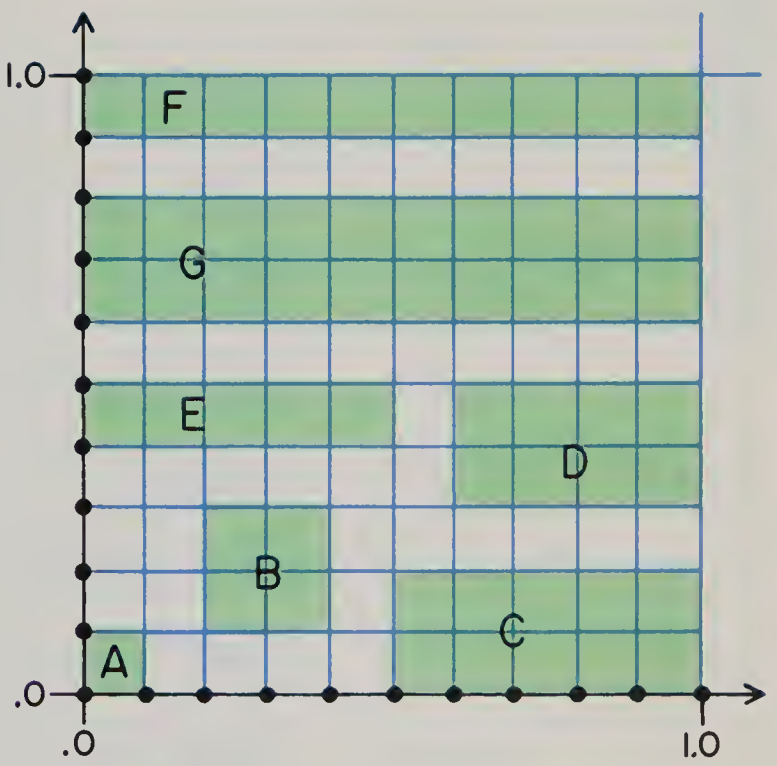
1×1 or $\frac{10}{10} \times \frac{10}{10}$ or 1.0×1.0 or $\frac{100}{100}$

$.4 \times .7$ or $\frac{4}{10} \times \frac{7}{10}$ or $\frac{28}{100}$

The area shaded blue suggests:

$.4 \times .7$ or $\frac{4}{10} \times \frac{7}{10}$ or $\frac{28}{100}$

In the diagram on the right, seven areas are shaded. What do they suggest?



Sketch: Suggests that:

A. $.1 \times .1 = \frac{1}{100}$

B. $.2 \times .2 = \frac{4}{100}$

C. $.5 \times .2 =$

D. $\_\_\_ \times \_\_\_ =$

E. $\_\_\_ \times \_\_\_ =$

F. $.1 \times 1.0 = \frac{10}{100}$

G. $\_\_\_ \times \_\_\_ =$

Instead of writing 100 so many times as a denominator, they would probably expand on their shorthand. Do you see their system?

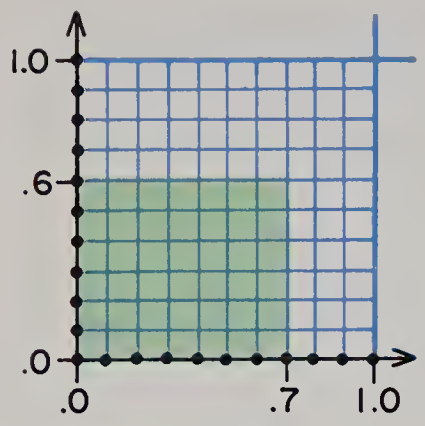
$.1 \times .1 = .01$ $\quad .1 \times 1.0 = .10$

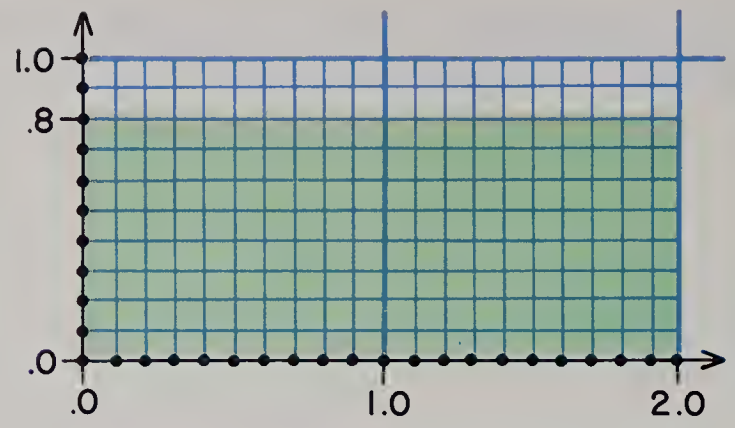
$.5 \times .2 = .10$ $\quad .80 + .27 = 1.07$

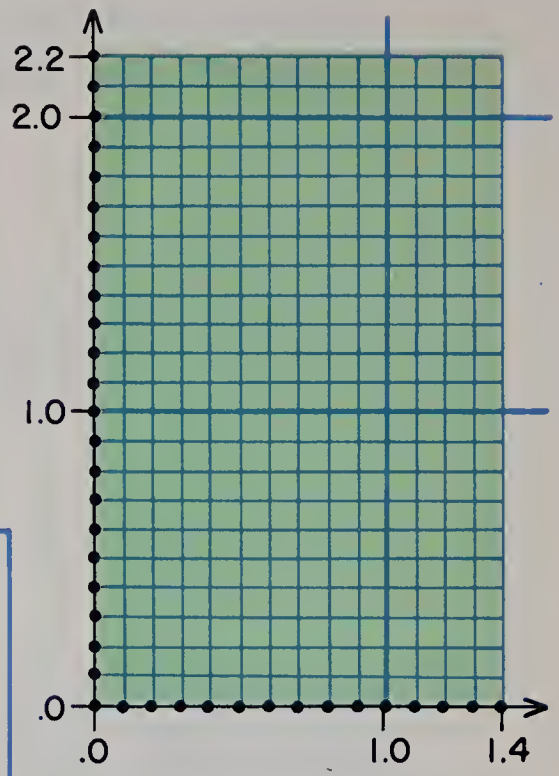
$.2 \times .4 =$ $\quad 1.0 \times 1.0 =$ $\quad .1 + 1.0 =$ $\quad .40 + .85 =$

$2.0 \times .2 =$ $\quad 1.0 \times 2.0 =$ $\quad .01 + .09 =$ $\quad 4.1 + .5 =$

$.1 \times .5 =$ $\quad 1.5 \times 1.0 =$ $\quad .01 + .1 =$ $\quad 4.1 + .05 =$







.7 x ___ = ___

2.0 x ___ = ___

1.4 x 2.2 = ___

1	$\frac{10}{10}$	1.0	$\frac{100}{100}$	1.00
8	$\frac{80}{10}$	8.0	$\frac{\quad}{100}$	8.00
10	$\frac{100}{10}$		$\frac{\quad}{100}$	10.00
$\frac{1}{2}$	$\frac{\quad}{10}$		$\frac{\quad}{100}$	
$\frac{1}{5}$	$\frac{\quad}{10}$		$\frac{\quad}{100}$	
$\frac{9}{30}$	$\frac{\quad}{10}$		$\frac{\quad}{100}$	

$\frac{2}{5}$	.4	.40
$1\frac{1}{2}$	___.5	
$2\frac{2}{5}$		
$10\frac{4}{20}$		
$\frac{28}{20}$		
$\frac{15}{50}$		
15		

$\frac{3}{6}$	$\frac{5}{10}$	.5 = .50
$\frac{18}{12}$	$\frac{\quad}{10}$	
$\frac{30}{4}$	$\frac{\quad}{10}$	

	I	II	III
a.	.3 + .4 = ___	.07 + .02 = ___	.3 + .2 = ___
b.	.6 + ___ = 1.0	.07 + ___ = .13	.30 + ___ = .50
c.	1.0 − .4 = ___	___ + .50 = .75	.3 + .20 = ___
d.	.8 + ___ = 1.6	1.50 − ___ = .95	.3 + .05 = ___
e.	2 x .8 = ___	2 x .75 = ___	.07 + ___ = .10
f.	1.6 ÷ ___ = .8	2 x ___ = 1.00	.7 + ___ = .73
g.	1.6 ÷ ___ = 2	___ x .09 = .27	.30 − .25 = ___
h.	___ − .6 = .3	.18 ÷ 2 = ___	.4 − .37 = ___
i.	9.0 − ___ = 8.4	.18 ÷ ___ = 3	.80 − ___ = .72
j.	9.0 − 6.0 = ___	1.00 − ___ = .90	.7 − .07 = ___
k.	___ x .7 = .7	___ + .10 = .11	2.4 + .12 = ___
l.	10 x .7 = ___	8 ÷ 100 = ___	1.8 − ___ = 1.75
m.	7 ÷ 10 = ___	8 ÷ 200 = ___	.37 + 3.7 = ___

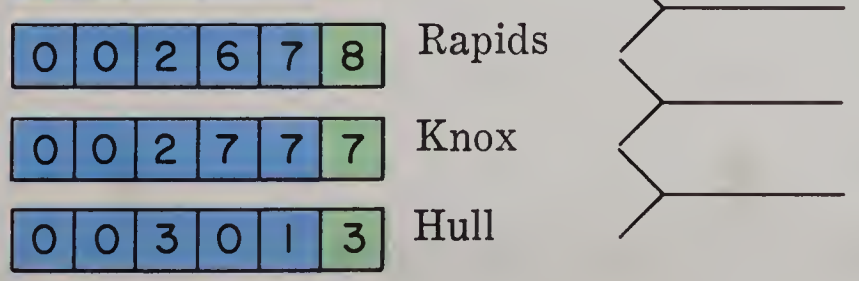
When we use this shorthand, we call that dot a "decimal point." We read it this way:

.3 — Three-tenths
.04 — Four-hundredths
1.7 — One and seven-tenths
3.12 — Three and twelve-hundredths

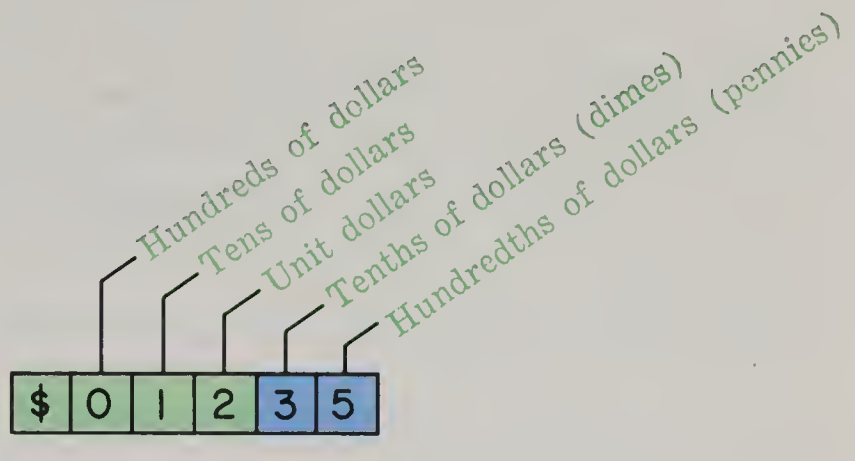
Instead of using a decimal point, a speedometer uses different colors.

At the start of a trip, the Moore family had a new car that had gone only seventeen and five-tenths miles.

Can you explain the record of their trip?


Tens of thousands of miles
Thousands of miles
Hundreds of miles
Tens of miles
Unit miles
Tenths of a mile
Distance in miles between towns
0 0 0 1 7 5 Home
0 0 0 2 7 5 Malta
10.0
0 0 0 4 7 9 Troy
0 0 0 5 0 0 Dover
0 0 0 8 5 2 Union
0 0 0 9 9 9 Lee City
0 0 1 0 8 0 Bureau
0 0 1 7 5 3 Olan
0 0 2 0 0 0 Benton
0 0 2 6 7 8 Rapids
0 0 2 7 7 7 Knox
0 0 3 0 1 3 Hull

A cash register often does not use a decimal point in the window that shows the amount of the sale. A different color is just as good.



Twelve dollars and thirty-five cents

Complete the chart below:

Amount of Sale	Money given clerk	Amount of change
$ 0 0 0 7 5	$1.00	$.25
$ 0 0 1 1 0	$5.00	
$ 0 1 2 5 0	$20.00	
$ 0 0 5 0 7	$10.00	
$	$5.00	$3.83
$ 0 0 0 5 8		$.17
$	$85.00	$8.02
$ 1 1 8 6 3	$120.00	
$ 0 1 0 8 0	$20.00	
$ 3 6 2 3 3	$500.00	
$	$94.37	$36.19
$ 0 2 7 4 6	$100.00	
$ 0 3 0 8 1		$17.69
$ 0 0 8 7 9	$10.00	
$	$25.00	$3.09
$ 4 8 6 2 9	$1,000.00	

TITLES or HEADLINES — On blue lines, use no shorthand. — On green lines, you may use shorthand.

APPROPRIATE PHRASES or STORIES

.1 — One dime . . . Six minutes . . .

360 seconds . . . $\frac{1}{10}$ of a mile . . .

.5 — Twelve hours . . . Fifty cents . . . Half-way . . .

30 minutes . . . $\frac{1}{2}$ off . . .

1.7 — One hour and forty-two minutes . . . Seventeen dimes . . .

$1.70 . . . 102 seconds . . . 1.7 gallons . . .

.6 , .2 — Six-tenths of a mile in two-tenths of an hour.

.9 , .3 → ___ — After we travelled nine-tenths of a mile, we still had three-tenths to go. That's one and two-tenth miles altogether.

1.6 , .4 → ___

.5 + .8 = ___

2 x 10.5 = ___

.25

2.50

1.30 , 2.8

3.18 ÷ 2 = _____

$$\begin{array}{r} 3.24 \\ 10.30 \\ +\ 6.68 \\ \hline \end{array}$$

Harry spent $4.32 for books that cost $1.08 each. He must have bought ____ books.

.6 x 2 = △

and

5 x △ = ____

A.

75) 52,650

B.

93 × 68

C.

685 × 105 = 000

— Check each example —

× 75

93) 6 4

Why do most people prefer this form?

685 × 105

105) 7 5

D.

307 × 209

307) 6 3

E.

28) 10,752

× 28

F.

617 × 57

57)

G.

804 × 95

5) 7 0

H.

38,194 × 173

173) 6 2

Rules for this DIVISION GAME are:

In ▭, write the largest whole number that makes sense in the example.

In ▭, write whatever is needed to make the statement true.

$8\overline{)56} = 8\overline{)56} + 8\overline{)0} = 7 + \frac{0}{8}$

$8\overline{)60} = 8\overline{)56} + 8\overline{)4} = \_\_ + \frac{1}{2}$

$10\overline{)78} = 10\overline{)} \quad 10\overline{)} = \_\_\_\_\_$

$12\overline{)154} = 12\overline{)} + 12\overline{)} = \_\_\_\_\_$

$25\overline{)180} = \overline{)} + \overline{)} = \_\_\_\_\_$

$16\overline{)106} = \overline{)} + \overline{)} = \_\_\_\_\_$

Where we "mix" a whole number and a fraction to be added, we usually omit the + sign.

$17 \div 10 = 1 + \frac{7}{10}$ or $1\frac{7}{10}$ or 1.7

$8 \div 7 = 1\frac{1}{7}$

$17 \div 6 = \_\_\_$ $\quad 28 \div 5 = \_\_\_$

$105 \div 10 = \_\_\_$ $\quad 100 \div 6 = \_\_\_$ $\quad 89 \div 25 = \_\_\_$ $\quad 160 \div 50 = \_\_\_$

$85 \div 9 = \_\_\_$ $\quad 350 \div 100 = \_\_\_$ $\quad 73 \div 4 = \_\_\_$ $\quad 565 \div 25 = \_\_\_$

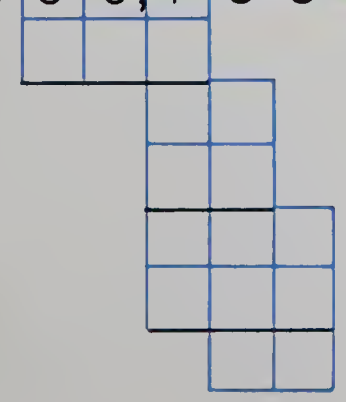
$5\overline{)3{,}483} = 5\overline{)} + 5\overline{)} = \_\_\_\_\_$

$10\overline{)9{,}878} = 10\overline{)} + 10\overline{)} = \_\_\_\_\_$

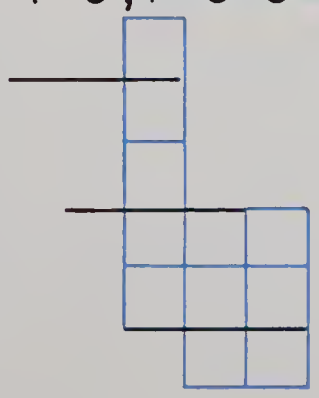
$93\overline{)6{,}335} = 93\overline{)} + \overline{)} = \_\_\_\_\_$

(See example B on page 74.)

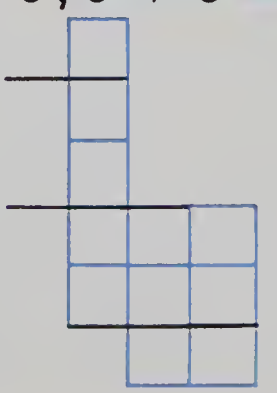
$57\overline{)3{,}518}$ —



$28\overline{)10{,}766}$ $\frac{2}{}$



$36\overline{)9{,}915}$ $\frac{12}{}$



In division, we sometimes use "R" as an abbreviation for "remainder."

There are two headlines below. Also, two situations are given. Which headline is most appropriate for each situation?

$17 \div 5 = 5\overline{)15} + 5\overline{)2} = 3 + \frac{2}{5}$ or $3\frac{2}{5}$

$17 \div 5 = 15 \div 5$ and R2 or 3^{R2}

A. Mrs. Brown had 17 house plants. She wanted to divide them among her 5 children so each would have the same number. After she had given 3 to each, she had 2 plants left.

B. Mr. Black had \$17.00 to distribute equally among his 5 children. After he had given each \$3.00, he had \$2.00 left. He distributed 40c to each. Each child received a total of $3\frac{2}{5}$ dollars; or \$3.40 each.

Make up stories of your own that involve these two kinds of "remainders."

List numbers which —

have remainders of	when divided by	a	b	c	d	e	f	g	h	i	j
2	5	17	32	2	7						
4	5										
3	4										
1	3										
1	2										
1	6										
	9	22	49	67							
	7	20	76	13							
8		30	19	52							
	8	69	101								
7		37	17	67							

List numbers which satisfy both of two conditions:

(Take your time, but keep an eye open for patterns that help.)

have remainders of	when divided by	AND have remainders of	when divided by	A	B	C	D	E	F	G
1	2	1	3							
2	3	2	6							
3	5	3	4							
2	10	1	2							
7	10	1	6							
3	4	3	9							
5	7	5	11							

(Same rules as above)

(Find a pattern, and then write examples as fast as you can.)

	R	when ÷	AND R	when ÷	H	I	J	K	L	M	N	O
I	1	2	1	5								
II	2	3	2	4								
III	4	6	4	7								
IV	3	5	3	8								
V	6	10	6	9								
VI	1	3	1	5								
VII	1	3	2	5								
VIII	1	3	3	5								
IX	1	3	4	5								
X	1	4	1	7								
XI	2	4	1	7								
XII	3	4	1	7								
XIII	1	9	2	4								
XIV	3	9	2	4								
XV	5	9	2	4								

Mr. Boles sold floor tile. Some of his tiles were 1 foot square — 12 inches by 12 inches — and were sold only in packs of 1 dozen each.

To save time, Mr. Boles made a chart so he could quickly answer questions as soon as he knew the length of a room to be covered with tile. Fill in the blanks in the part of his chart shown below:

(In chart below, all rooms are more than 4 feet wide.)

Size of room		No. of tiles needed	No. of packages needed	No. of tiles not used
length	width			
7 ft.	5 ft.	35	3	1
7 ft.	6 ft.			
7 ft.	7 ft.			
7 ft.	8 ft.			
7 ft.	9 ft.			

Very Hard →

Size of room		No. of tiles needed	No. of packages needed	No. of tiles not used
length	width			
		55		
			7	9
	15	225		
14			19	4
	7			5

Problem: Arrange a given number of tiles in a rectangular array (even rows and columns). Always use as many tiles as possible!

EXAMPLE ONE: You have 84 pieces of tile.

	a	b	c	d	e	f	g	h	i	j	k	l
No. of rows	8		15	5				6	17		29	
No. of columns	10	12			4	9	19			11		23
No. of tiles used	80							78				
No. of tiles not used	4								16	7		

EXAMPLE TWO: You have 156 pieces of tile.

	m	n	o	p	q	r	s	t	u	v	w	x
No. of rows	15	20			9		30				33	
No. of columns	10		24	13		28		37	19			5
No. of tiles used												
No. of tiles not used	6									2		

	A	B	C	D	E	F	G	H	I	J	K
Change no. of days shown	10	17	23		28		6	0	31		
to largest no. of weeks	1	2		2		3				11	14
and smallest no. of days	3			1		5				4	6

When a number is divided by 7, the remainder is 0, 1, 2, 3, 4, 5 or 6.

In column A below, list all numbers that leave a remainder of 0 when divided by 7. In column B, list those leaving a remainder of 1 when divided by 7.

(Each column should list numbers from smallest to largest.)

When divided by 7, the remainder (0) (1) (2) (3) (4) (5) (6)

Column → (A)	(B)	(C)	(D)	(E)	(F)	(G)
0	1					6
7		9				
14						
21						

In the blue brackets, indicate the remainder when each number is divided by 7.

( )	( )	(6)	( )	( )	( )
9 + 18 = ___			10 + 24 = ___		
(6)	(1)	( )	( )	( )	( )
___ + ___ = 28			2 x 10 = ___		
( )	( )	( )	(3)	(0)	( )
3 x 11 = ___			___ x ___ = ___		

Under the letters indicated, use any number that appears in that column or would appear in that column if the chart were extended.

(C) (D) ( )	(B) (G) ( )
9 + 17 = ___	8 + 13 = ___
___ + ___ = 33	___ + ___ = 14
16 + ___ = ___	22 + ___ = ___
___ + 10 = ___	___ + ___ = ___
(E) (E) ( )	**(D) (A) ( )**
18 + 4 = ___	24 + ___ = ___
___ + ___ = ___	___ + ___ = ___
___ + ___ = ___	___ + ___ = ___
___ + ___ = ___	___ + ___ = ___
(B) (F) ( )	**(C) (F) ( )**
1 x 5 = ___	2 x 5 = ___
8 x 5 = ___	2 x 19 = ___
___ x ___ = ___	___ x ___ = ___
___ x ___ = ___	___ x ___ = ___
(A) (E) ( )	**(C) (A) ( )**
7 x ___ = ___	___ x 0 = ___
0 x ___ = ___	___ x 21 = ___
14 x ___ = ___	___ x 28 = ___

+	A	B	C	D	E	F	G
A							
B							
C				F			
D			F				
E							
F							
G							

On chart at left we have shown that any number in column C added to any number in column D gives a number somewhere in column F in the table above. Can you fill in the remaining spaces?

On chart at right are shown examples of what happens when we multiply. Fill in the remaining spaces if you can.

X	A	B	C	D	E	F	G
A							
B							
C				G			
D			G				
E							
F							
G							

Another setting on Alec's machine gave these results.

Find the pattern and complete "Machine writes"

You write	Machine writes		
	$< \frac{1}{4}$	$\frac{1}{4}$	$> \frac{1}{4}$
$\frac{1}{4}$ $\frac{1}{3}$ $\frac{25}{100}$ $\frac{2}{10}$ $\frac{3}{8}$ 17%	17%	$\frac{1}{4}$	.5
50% 18 100% $\frac{29}{100}$ $\frac{23}{100}$	$\frac{23}{100}$ $\frac{1}{5}$	$\frac{3}{4} - \frac{1}{2}$ 25%	18 $\frac{1}{3}$
$\frac{3}{4} - \frac{1}{2}$ $\frac{3}{10}$ .18 25% .09			
.25 5% $\frac{5}{8} - \frac{1}{2}$ 2.1 .5 $\frac{25}{10}$ $\frac{1}{5}$			

To complete the record, you need to know what the shorthand means that uses "%." That is another sign for "hundredths." Maybe it had a history like this:

$\frac{25}{100}$ → $\frac{25}{0/0}$ → $\frac{25}{\%}$ → 25%

So, we can say that —

$\frac{25}{100} = \frac{1}{4} = .25 = 25\% = \frac{2}{8} = \frac{5}{20}$ etc.

$\frac{50}{100} = \frac{}{2} = .50 = .5 = \_\_\% = \frac{}{30}$

$\frac{1}{5} = \frac{}{10} = \frac{}{100} = \_\_\% = .\_\_ = \frac{}{15}$

$\frac{3}{4} = \frac{}{40} = \frac{}{100} = \_\_\% = .\_\_ = \frac{}{16}$

Make the following true, using

= means "equals"
< means "less than"
> means "greater than"

$\frac{1}{2}$ ▢ 50%	$\frac{1}{4}$ ▢ 30%	10% ▢ $\frac{10}{100}$
$\frac{1}{3}$ ▢ 25%	$\frac{2}{5}$ ▢ 40%	75% ▢ .8
90% ▢ $\frac{90}{100}$	100% ▢ $\frac{100}{100}$	100% ▢ 1
125% ▢ $1\frac{1}{4}$	150% ▢ $1\frac{3}{4}$	200% ▢ $1\frac{1}{2}$
23% ▢ $\frac{1}{4}$	$\frac{3}{4}$ ▢ 72%	1% ▢ .1
15% ▢ $\frac{3}{20}$	$\frac{1}{4}$ ▢ 30%	1% ▢ $\frac{1}{2}$

Here are some common uses of this new shorthand for an old idea:

25% off on all items.

We pay 3% on all savings deposits.

We won 50% of our games.

35% Dacron, 65% cotton.

100% Beef Frankfurters.

Pay only 75% of marked price.

What do you think these expressions mean?

Below, write sentences that include the following expressions:

50% 10% 4% 125%

Fill in the following as true statements. Do not repeat the same idea.

50 % of 38 = ____	25 % of ____ = 7	____ % of 4 = 3
8 = 10 % of ____	____ = 20 % of ____	____ = 5 % of ____
____ % x 36 = 9	____ = ____ % x 40	75 % x ____ = ____
____ = 15 % x ____	150 % x ____ = ____	12 = ____ % x 8
____ = ____ % x ____	____ = ____ % x ____	____ % x 20 = 4

Each box below is to contain a part of the amount on the left. That part is indicated at the top of each column.

	50%	25%	10%	75%	5%	150%
1 hour	min.	min.	min.	min.	min.	min.
1 yard and 4 inches	in.	in.	in.	in.	in.	in.
5 dollars	$	$	$	$	$	$

	10%	____%	15%	110%	130%	____%
$ 25.00	$	$ 5.00	$	$	$	$ 62.50
____ minutes	sec.	sec.	sec.	sec.	sec.	5 min.
____ feet	1 ft.	ft.	in.	ft.	ft.	ft.

Write an appropriate story for each of the three headlines given below :

40 % x 25 = ____

7 = 10 % x ____

2.25 = ____ % of 9

Each group of numbers in the tinted area is written without order. Rearrange each group on the right as indicated by the signs.

Remember: $<$ means "is less than"
$>$ means "is greater than"
$=$ means "is equal to"

a.	$\frac{1}{2}$	75%	.4	$\frac{8}{10}$	.49	41%	<	<	<	<	<
b.	.7	125%	1	$\frac{16}{10}$	85%	1.5	>	>	>	>	>
c.	$\frac{1}{2}$	.03	3	.3	30	$\frac{1}{4}$	<	<	<	<	<
d.	1.48	$\frac{9}{10}$	99%	150%	$1\frac{1}{100}$	$1\frac{1}{2}$	=	>	>	>	>
e.	$\frac{3}{10}$	20%	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$	60%	=	>	>	>	=
f.	1.00	$\frac{6}{3}$	200%	125%	1.9	$\frac{10}{8}$	<	=	<	<	=
g.	39%	$\frac{9}{20}$	$\frac{3}{100}$	.49	2%	$\frac{2}{5}$	<	<	<	<	<
h.	$\frac{1}{8}$	1.0	6	$\frac{8}{2}$	$\frac{4}{36}$	12%	>	>	>	>	>
i.	$\frac{7}{20}$	.06	.35	06%	$\frac{3}{50}$	35.0%	=	=	>	=	=
j.	.75	1000%	50%	10	$\frac{15}{20}$	$\frac{11}{22}$	=	<	=	<	=
k.	$\frac{11}{100}$	$\frac{7}{50}$	13%	$\frac{3}{25}$	.1	9%	<	<	<	<	<
l.	$\frac{1}{20}$	$\frac{1}{100}$	$\frac{5}{2}$	5%	3	250%	>	=	>	=	>

Using $<$, $>$ or $=$, make three true comments about each expression:

25% x 7	1	2	1.5	$3 \times \frac{11}{10}$	2	7	10
10% x 83	8.00	80.0	800	3 x .2	$\frac{6}{1}$	$\frac{6}{10}$	$\frac{6}{100}$
1% x 97	10	1	.7	.3 x .2	$\frac{6}{1}$	$\frac{6}{10}$	$\frac{6}{100}$
.10 x 25	.20	2.0	20	.5 x .7	35	3.5	.35
$\frac{10}{11} + \frac{19}{20}$	.5	2.5	1	8 x .4	32	3.2	.32
.20 x 37	.6	7	18	1.2 x 1.2	1	2	14.4
$\frac{1}{3} + \frac{1}{2}$	$\frac{1}{6}$	1	5	1.0 x 9	.90	.09	9.0
$\frac{7}{10} - \frac{30}{100}$	$\frac{1}{7}$	.4	2.1	.3 x 25	7.5	$\frac{3}{8}$	75%
7 ÷ 2	34	.4	4	5 x .2	1%	10%	100%
.7 ÷ 2	.3	5	1.4	.8 x .5	4.0	.04	.4
7 ÷ $\frac{1}{2}$	1.4	14	35	.3 x .3	1.0	0.1	.01

Complete sentences below. (Do not simplify any of the fractions.)

$\frac{6}{10} \times \frac{8}{1} = \frac{\quad}{10}$

$\frac{3}{10} \times \frac{9}{100} = \frac{\quad}{1{,}000}$

$\frac{24}{100} \times \frac{2}{10} = \frac{\quad}{\quad}$

$\frac{4}{100} \times \frac{12}{10} = \frac{\quad}{\quad}$

$\frac{12}{100} \times \frac{9}{100} = \frac{\quad}{\quad}$

$\frac{1}{10} \times \frac{13}{1} = \frac{\quad}{\quad}$

$\frac{\quad}{100} \times \frac{\quad}{100} = \frac{49}{1{,}000}$

$\frac{\quad}{10} \times \frac{7}{\quad} = \frac{84}{1{,}000}$

$\frac{4}{100} \times \frac{\quad}{100} = \frac{32}{\quad}$

$\frac{1}{10} \times \frac{\quad}{100} = \frac{57}{\quad}$

$\frac{13}{\quad} \times \frac{\quad}{10} = \frac{156}{10}$

$\frac{24}{100} \times \frac{15}{10} = \frac{\quad}{\quad}$

$\frac{2}{10} \times \frac{4}{10} = \frac{\quad}{100}$

$\frac{5}{100} \times \frac{5}{100} = \frac{\quad}{10{,}000}$

$\frac{8}{1} \times \frac{8}{10} = \frac{\quad}{\quad}$

$\frac{2}{10} \times \frac{9}{10} = \frac{\quad}{\quad}$

$\frac{12}{10} \times \frac{12}{1} = \frac{\quad}{\quad}$

$\frac{9}{10} \times \frac{\quad}{\quad} = \frac{81}{100}$

$\frac{\quad}{10} \times \frac{7}{\quad} = \frac{35}{10}$

$\frac{11}{\quad} \times \frac{\quad}{10} = \frac{55}{100}$

$\frac{24}{100} \times \frac{57}{\quad} = \frac{\quad}{10{,}000}$

$\frac{9}{100} \times \frac{2}{10} = \frac{\quad}{\quad}$

$\frac{45}{10} \times \frac{57}{100} = \frac{\quad}{\quad}$

$\frac{12}{100} \times \frac{18}{100} = \frac{\quad}{\quad}$

Record the results from each example on the left. Then complete the charts.

X	8	3	7	12
.6	4.8	1.8		
.8	6.4			
.5				
1.0				
1.2				

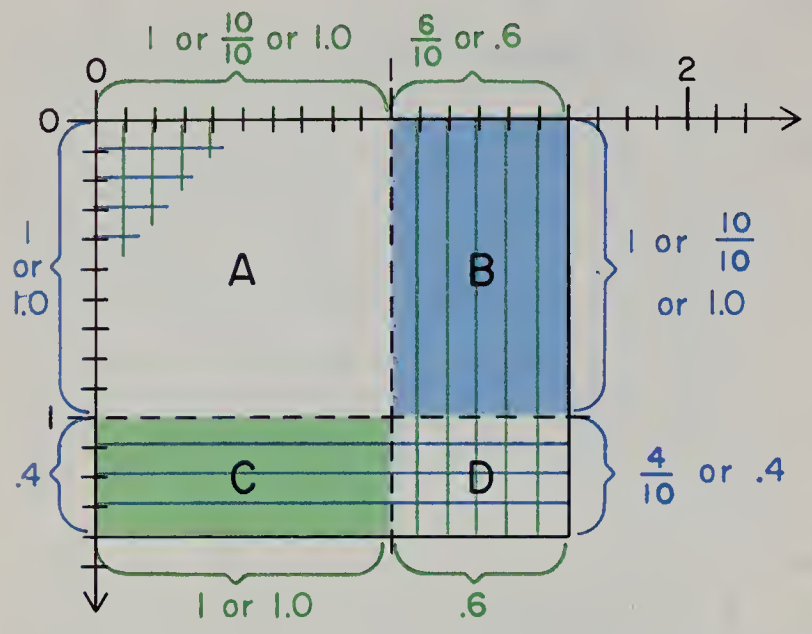
X	.4	.6	.9	.5
.2	.08			
.7				
.4				
.9				
1.1				

X	.3	.8	1.2	.7
.09				
.04				
.11				
.07				
.03				

X	.2	13	1.5	.57	.18
.24					
.1					
4.5					
.09					
1.2					

X	.05	.09	.04	.06
.05				
.10				
.06				
.08				
.12				

Familiar pictures can help explain the idea of multiplying decimal fractions.



"→" means "suggests that" or "is a picture that suggests that"

A → 1 x 1 = 1 and 1.0 x 1.0 = 1.00

B → 1 x .6 = .6 and 1.0 x .6 = .60

C → .4 x 1 = .4 and .4 x 1.0 = .40

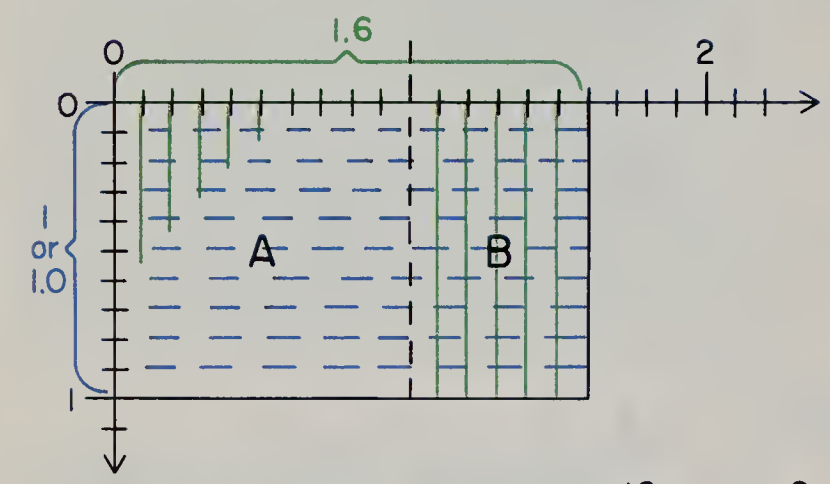
D → .4 x .6 = _____

A & B → (1 x 1) + (1 x .6) = 1.6

A & C → (1 x 1) + (.4 x 1) = _____

C & D → (.4 x 1) + (.4 x .6) = _____

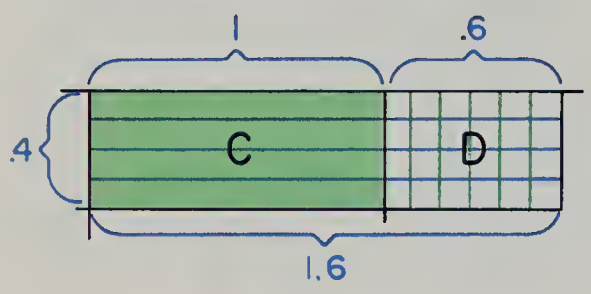
A & B & C & D → (1 x 1) + (1. x .6) + (.4 x 1) + (.4 x .6) = _____



A & B → 1 x 1.6 = 1.6 or 16/10 or 1 6/10

and

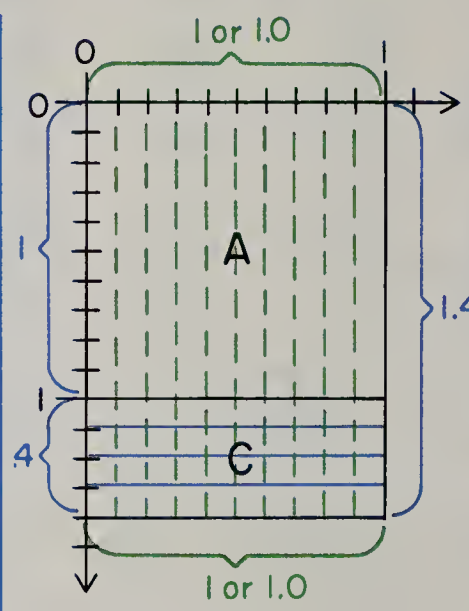
A & B → 1.0 x 1.6 = 1.60 or 160/100 or 1 60/100



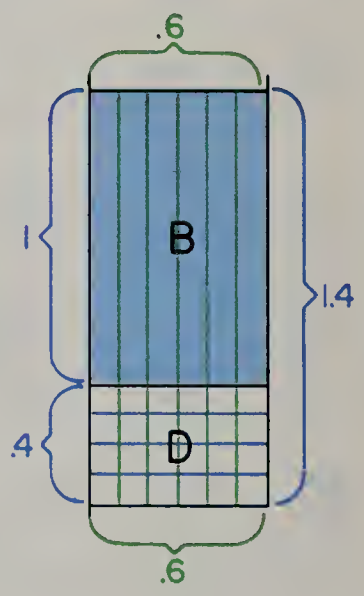
C & D → .4 x 1 + .4 x .6 = .4 + .24 = .64

and

C & D → .4 x 1.6 = .64



A & C →
(1 x 1) + (.4 x 1) =
1 + .4 = 1.4 =
1.00 + .40 = 1.40



B & D →
1 x .6 + .4 x .6 =
.6 + .24 =
.60 + .24 = .84

These results look familiar. We already know that . . . . .

	1	6
x	1	4

	1	4
x	1	6

C & D → .64

A & B → 1.60

A & B & C & D → 2.24

1.6 x 1.4 = 2.24

B & D → .84

A & C → 1.40

A & B & C & D → 2.24

1.4 x 1.6 = 2.24

ON THIS PAGE . . . . .

This size is a picture to suggest that

1 x 1 = 1
1 x 1.0 = 1.0
1.0 x 1.0 = 1.00

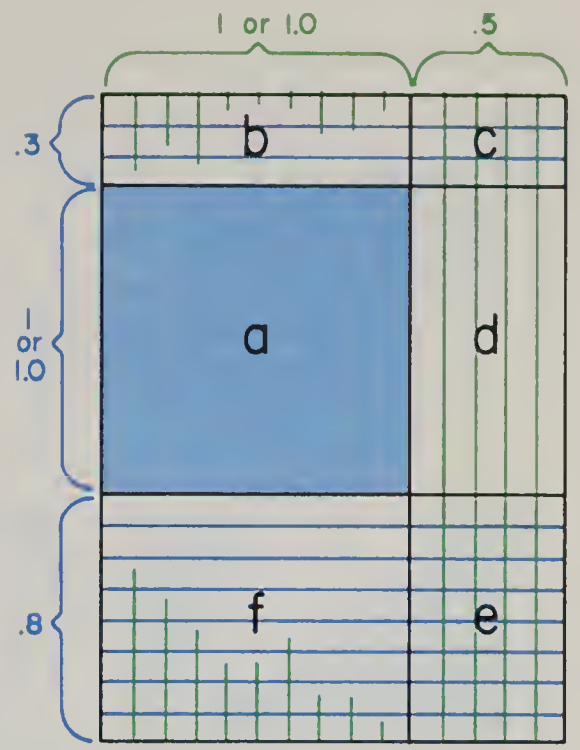
a → 1 x 1 = 1

b → .3 x 1 = .3

c → .3 x .5 = .15

d → _____ = ___

e → _____ = ___



b a c → .3 x (1.0 + .5) = .3 x 1.0 + .3 x .5 = ___ + ___ = ___

f a e → .8 x (1.0 + .5) = .8 x 1.0 + .8 x .5 = ___ + ___ = ___

d a e → (1.0 + .8) x .5 = 1.0 x .5 + .8 x .5 = ___ + ___ = ___

---

a a b → ___ x ___ = ___ + ___ = ___
(a, b)

a a f → ___ x ___ = ___ + ___ = ___
(a, f)

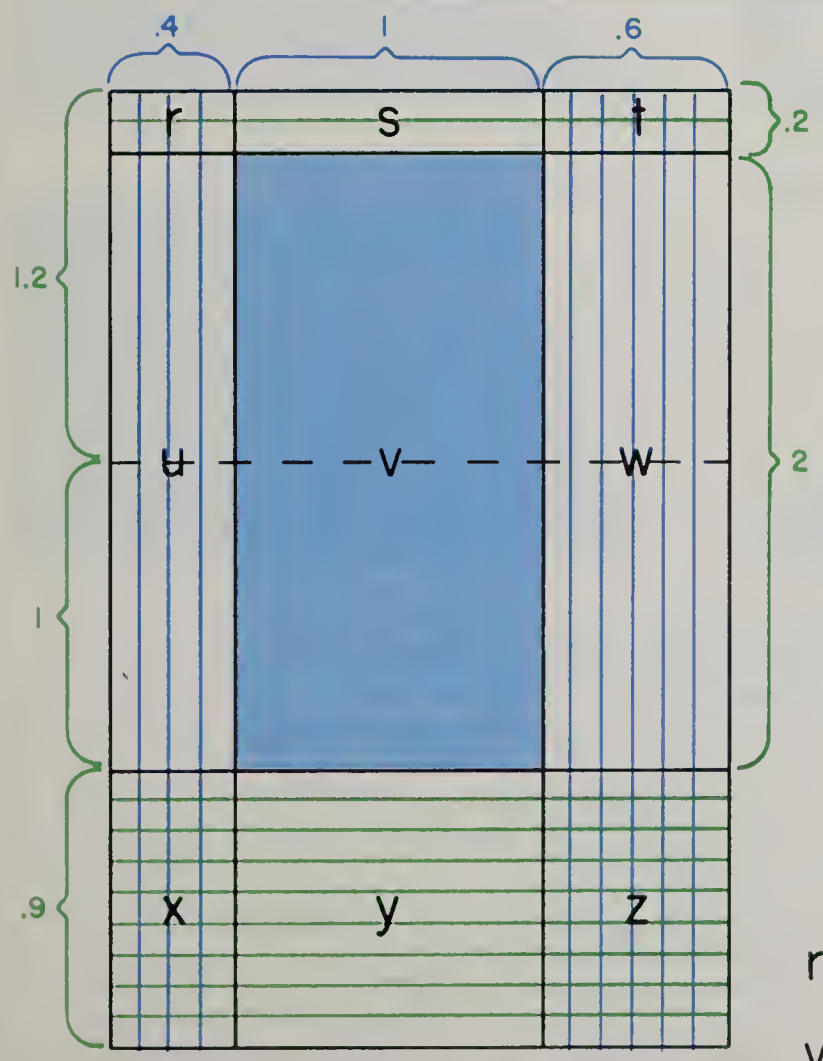
c a d → ___ x ___ = ___ + ___ = ___
(c, d)

a a d → ___ x ___ = ___ + ___ = ___
(a, d)

a a b a c a d → ___ x ___ = ___ + ___ + ___ + ___ = ___
(a, b, c, d)

a a f a d a e → ___ x ___ = ___ + ___ + ___ + ___ = ___
(a, f, d, e)

a a b a c a d a e a f → ___ x ___ = ___ + ___ + ___ + ___ + ___ = ___
(a, b, c, d, e, f)



r → x = ___ S → x = ___

t → x = ___ W → x = ___

z → x = ___ X → x = ___

u → x = ___ V → x = ___

x a y → ___ x ___ = ___ + ___ = ___

v a w → ___ x ___ = ___ + ___ = ___

w a z → ___ x ___ = ___ + ___ = ___

r a s a t → ___ + ___ + ___ = ___

s a v a y → ___ + ___ + ___ = ___

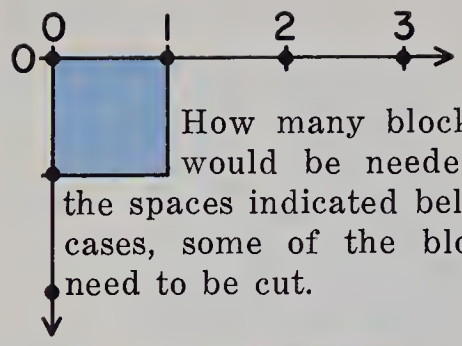
u a v a w → ___ + ___ + ___ = ___

r a s a u a v → ___ + ___ + ___ + ___ = ___

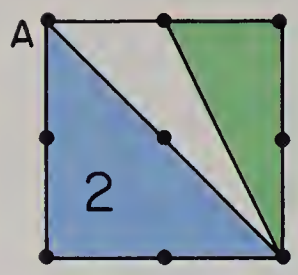
v a w a y a z → ___ + ___ + ___ + ___ = ___

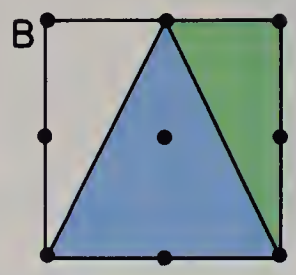
u a v a x a y → ___ + ___ + ___ + ___ = ___

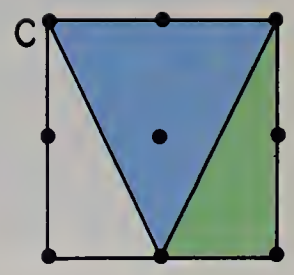
Indicate size of blue areas first.

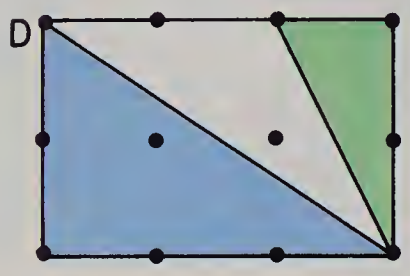


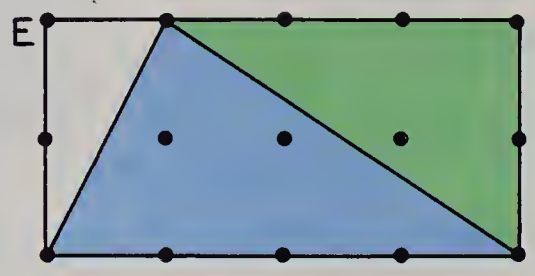
How many blocks this size would be needed to cover the spaces indicated below? In all cases, some of the blocks would need to be cut.

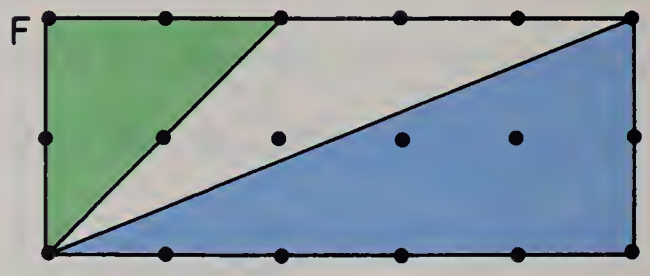


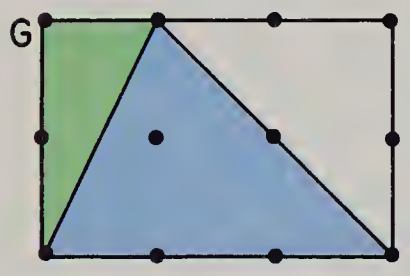


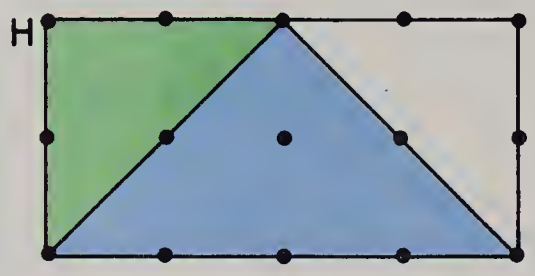


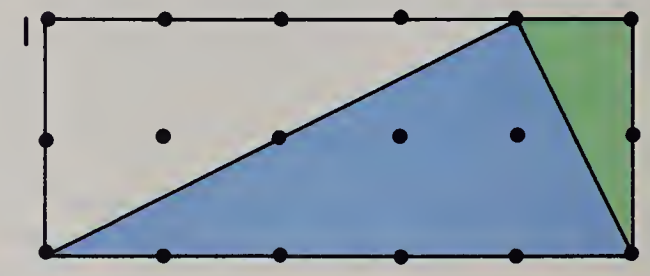


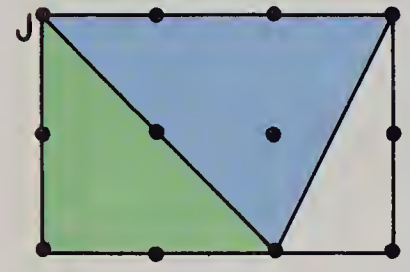


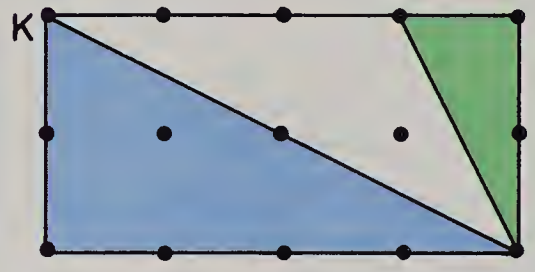


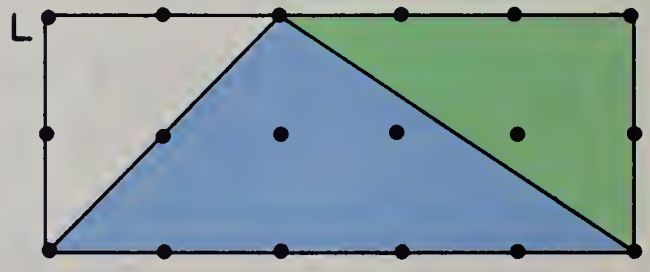


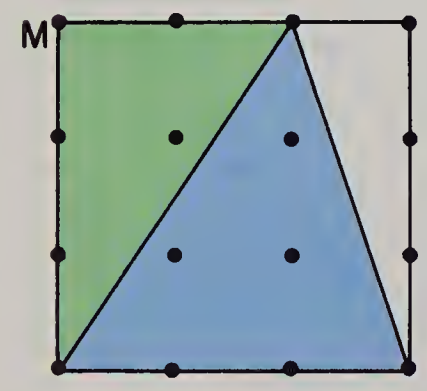


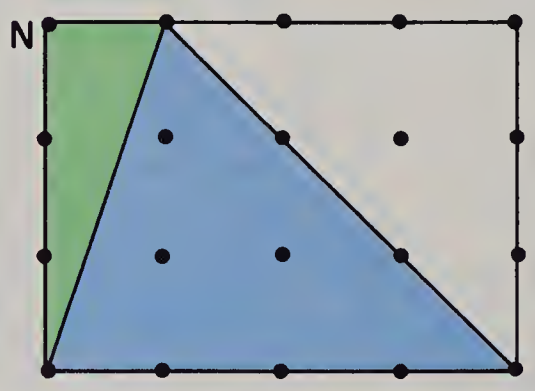


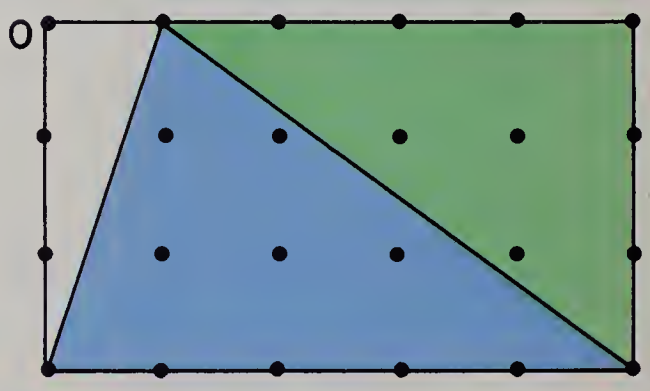




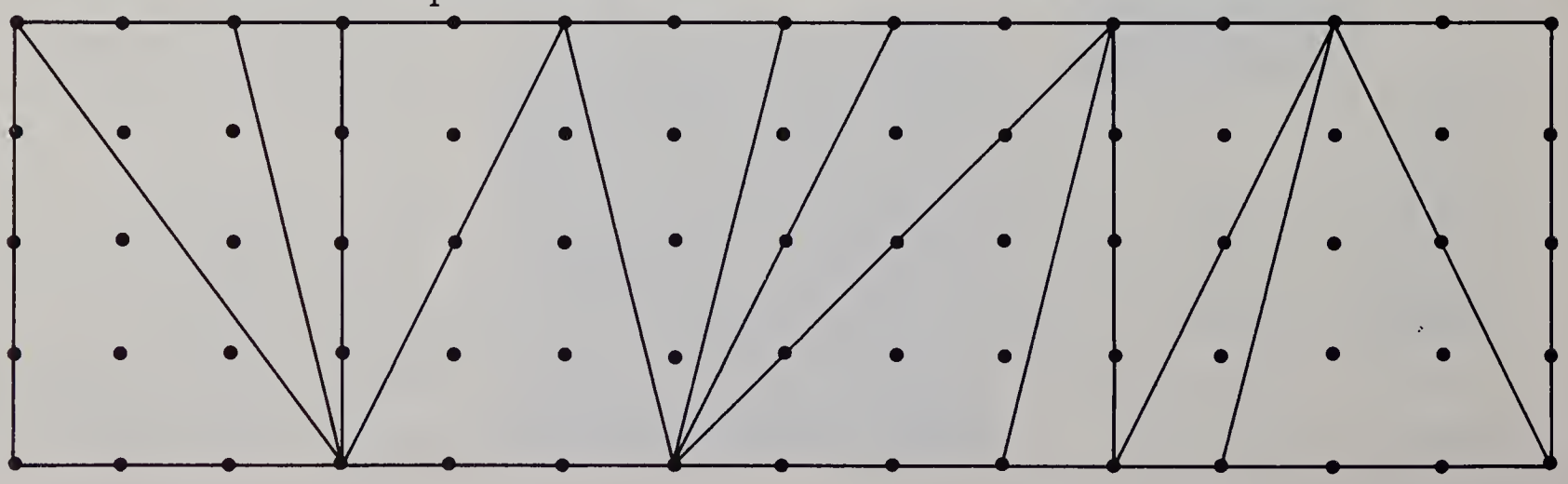




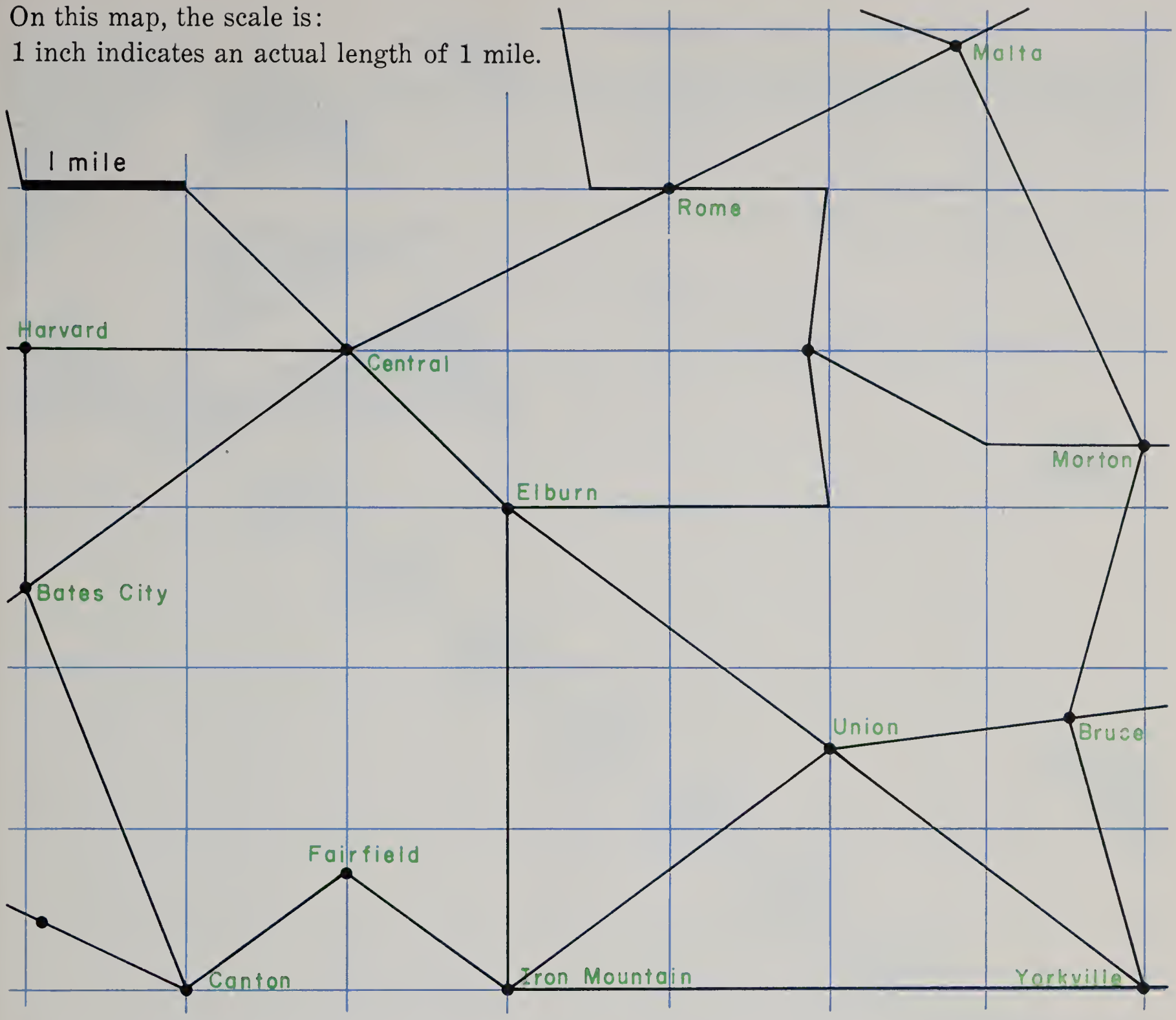




Color all shapes below that are the same size with the same color.



On this map, the scale is:
1 inch indicates an actual length of 1 mile.



What is the shortest distance by road between the towns given below? (Give distance as the nearest quarter of a mile.)

Harvard and Rome: _____
Bates City and Fairfield: _____
Morton and Iron Mountain: _____
Yorkville and Canton: _____
Morton and Rome: _____
Central and Yorkville: _____
Harvard and Fairfield: _____
Bates City and Malta: _____
Elburn and Canton: _____
Malta and Yorkville: _____
Elburn and Morton: _____

Mr. Drum is a salesman living in Bates City. He wants to leave home in the morning and call on customers in Fairfield, Central, Union and Bruce. What is the shortest distance he can travel, making all these stops and arriving at home in time for dinner?

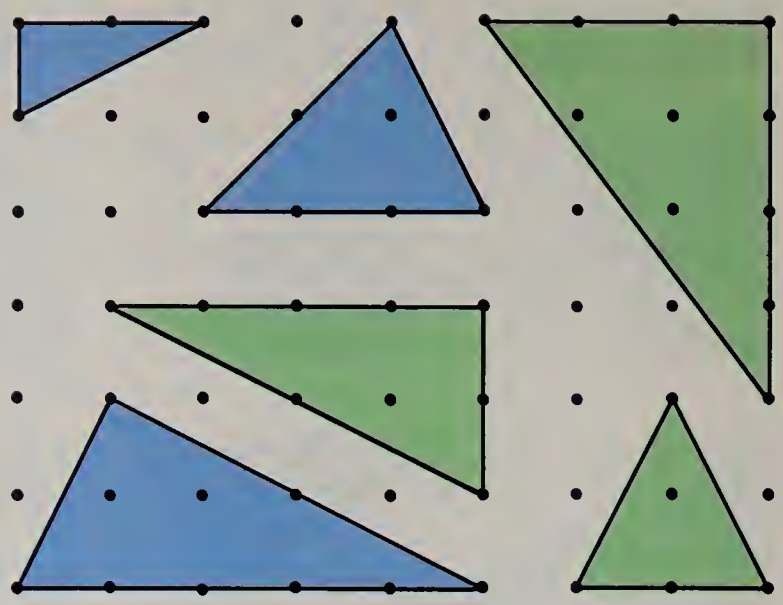
Find the length of the shortest trip by road from Morton to Rome by way of Fairfield.

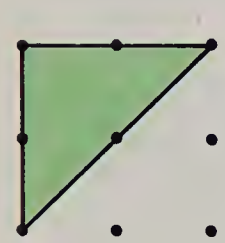
What is the shortest distance a crow would fly from Canton to Rome?

Which two cities shown are furthest apart as the crow flies? How far apart?

By connecting dots as in the examples given, draw at least three more triangles the same size as each example. Make all triangles different in shape. Two triangles are of a different shape if one, when it is cut out, could not be nicely fitted on the other without any cutting.

Color all triangles that are the same size in the same color — blue or green.





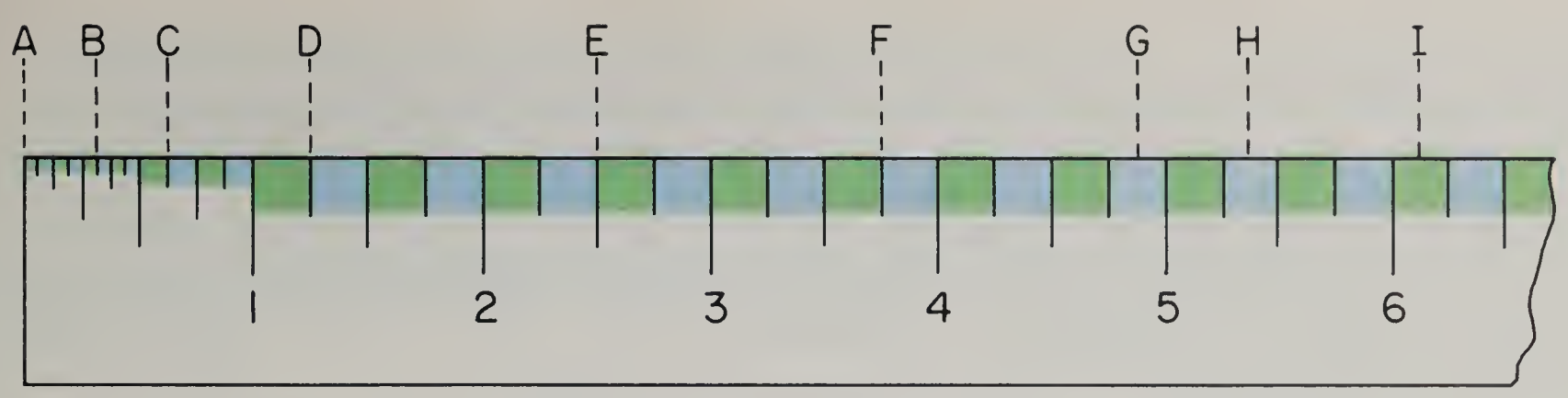


Chart distance in inches from . . . .

to... \ from...	A	B	C	D	E	F	G	H	I
A				$1\frac{1}{4}$	$2\frac{1}{2}$				
B									
C									
D						$2\frac{1}{2}$			
E									
F					$1\frac{1}{4}$				
G									
H									
I									

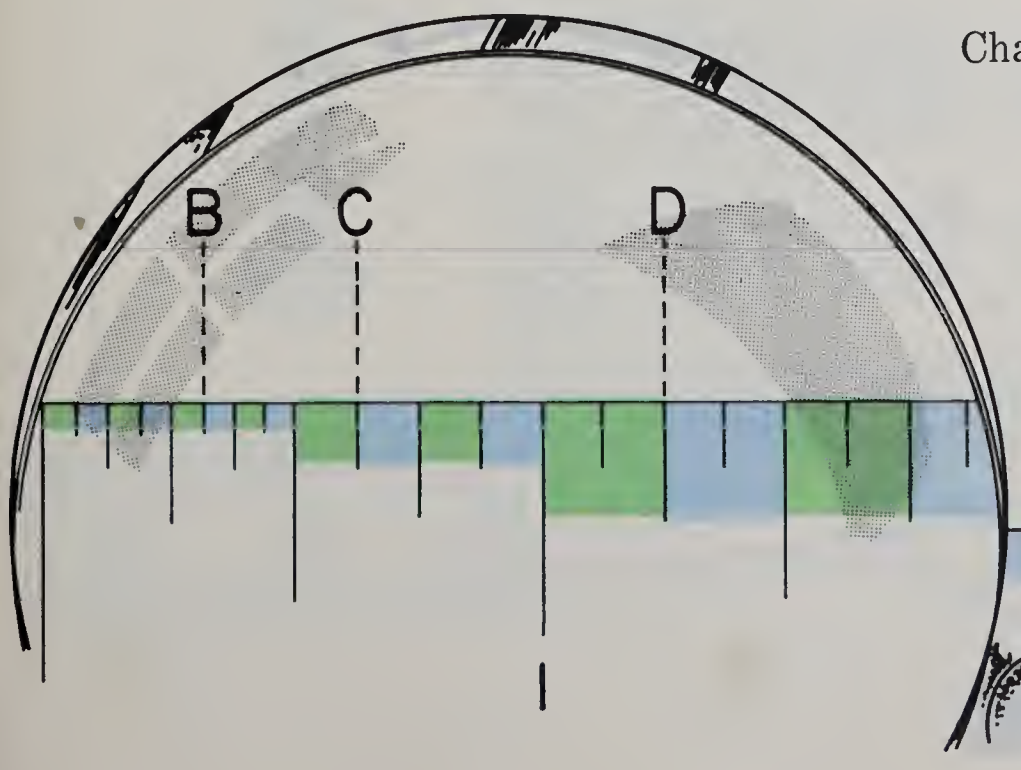
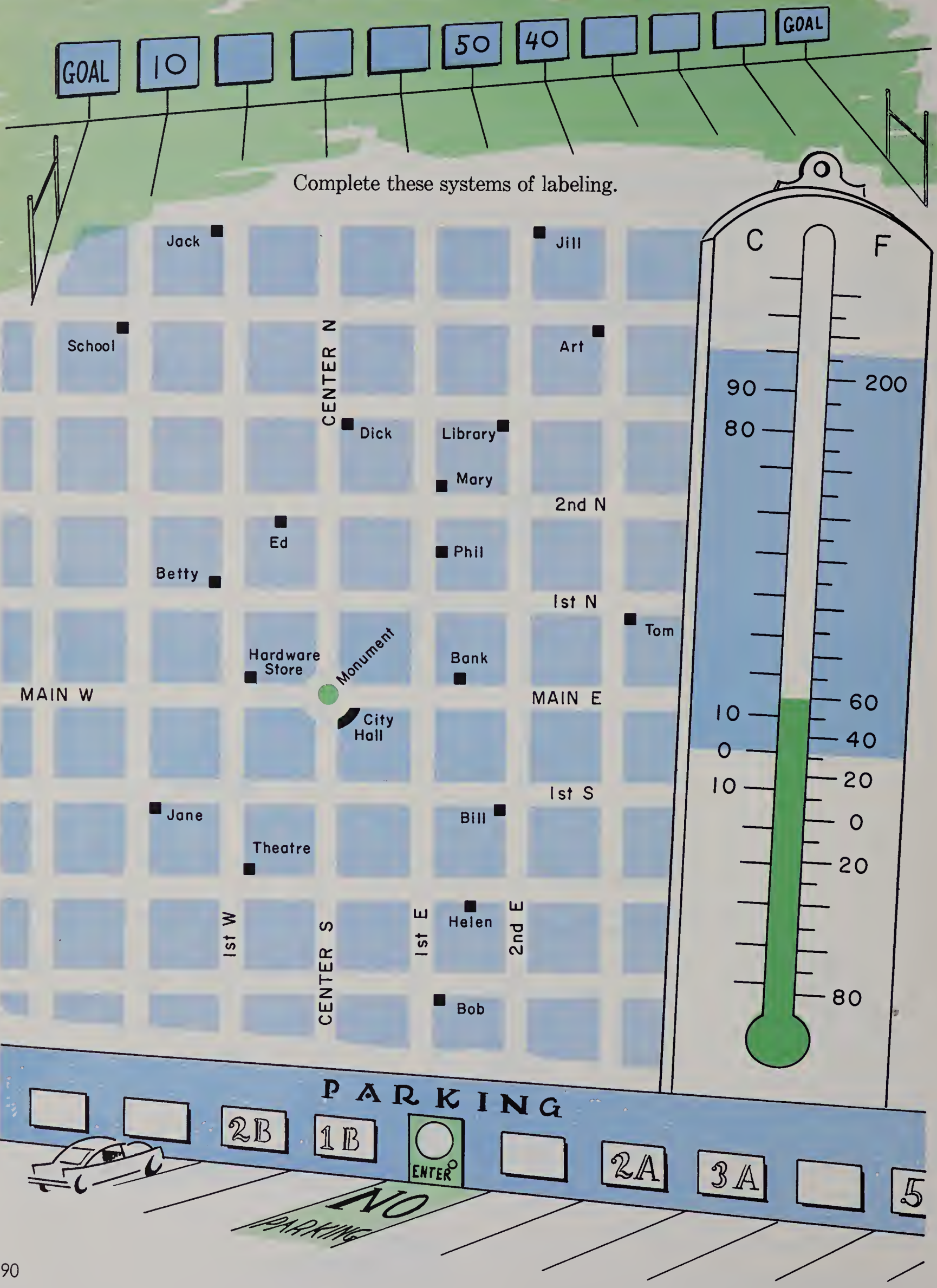


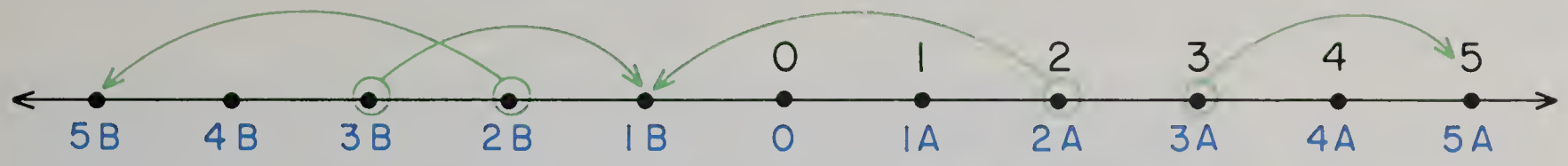
Chart in inches, $\frac{1}{2}$ of the distance from . . .

to... \ from...	A	E	G	H
C				
I				



Complete these systems of labeling.

The parking lot system and both scales on the thermometer suggest an extension of the Number Line system.



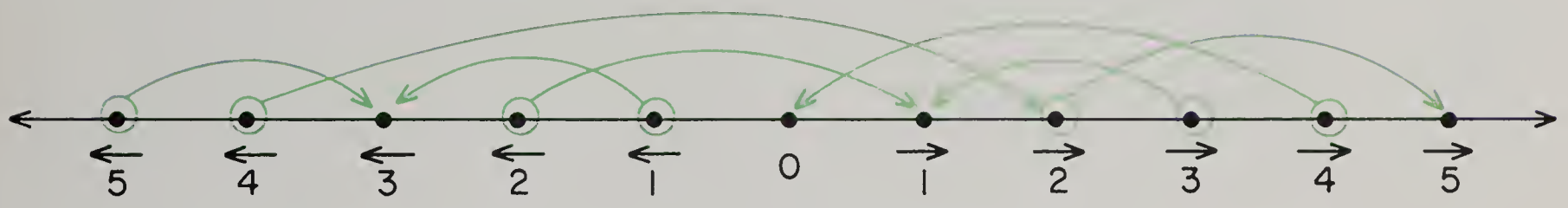
If you start at 2B and jump 3 spaces to the left, you land on 5B.

(read that larger arrow as "goes to")

2B, $\overleftarrow{3}$ ⟶ 5B 3B, $\overrightarrow{2}$ ⟶ 1B

2A, $\overleftarrow{3}$ ⟶ 3A, $\overrightarrow{2}$ ⟶

Let's mark the Number Line with labels that indicate the way you would get there if you started at 0.



Start at 0. Take a jump of 5 to the left, then take a jump of 2 to the right. You go to a point 3 to the left of 0.

$\overleftarrow{5}$, $\overrightarrow{2}$ ⟶ $\overleftarrow{3}$

$\overleftarrow{1}$, $\overleftarrow{2}$ ⟶ $\overleftarrow{3}$ $\overleftarrow{2}$, $\overrightarrow{3}$ ⟶ ---

$\overrightarrow{4}$, $\overleftarrow{4}$ ⟶ --- $\overrightarrow{2}$, $\overrightarrow{3}$ ⟶ ---

$\overrightarrow{3}$, $\overleftarrow{2}$ ⟶ --- $\overleftarrow{4}$ $\overrightarrow{6}$ ⟶ ---

First a jump of . . .

goes to ↘	$\overleftarrow{3}$	$\overleftarrow{2}$	$\overleftarrow{1}$	0	$\overrightarrow{1}$	$\overrightarrow{2}$	$\overrightarrow{3}$
$\overrightarrow{3}$		$\overrightarrow{1}$				$\overrightarrow{5}$	
$\overrightarrow{2}$							
$\overrightarrow{1}$							
0				0			
$\overleftarrow{1}$							
$\overleftarrow{2}$			$\overleftarrow{3}$				
$\overleftarrow{3}$							

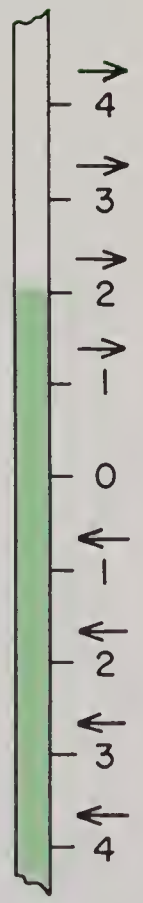
and then a jump of . . .

On the thermometer shown here, "$\rightarrow$" means above 0 and "$\leftarrow$" means below 0.

If the thermometer read . . .

↘	$\overrightarrow{5}$	$\overleftarrow{8}$	$\overleftarrow{21}$	$\overrightarrow{72}$	$\overrightarrow{60}$
rose 9°	$\overrightarrow{14}$	$\overrightarrow{1}$			
fell 15°	$\overleftarrow{10}$				
fell 11°					
rose 28°					
fell 16°					

and then . . .



In the headlines below, directional arrows are used. They can indicate many ideas. If "→" means "forward," then "←" will mean "backward." We can say that "→" is the opposite of "←", or that "←" is the opposite of "→". (Complete the pairs of ideas.)

If → means . . .	Up	Profit			Found	
then ← means . . .	Down		West	Below		

a $\overrightarrow{17}$, $\overleftarrow{25}$ ____

b $\overleftarrow{8}$, ____ $\overrightarrow{21}$

c $\overleftarrow{1.25}$, ____ $\overleftarrow{5}$

d , $\overleftarrow{4}$ $\overleftarrow{1\frac{1}{2}}$

e $\overrightarrow{5}$, $\overleftarrow{13}$, $\overrightarrow{15}$ ____

f Bill started down the basement stairs. After going down 8 stairs, he changed his mind. Then he went up 17 stairs.

g Harry lost $1\frac{1}{2}$ pounds one week and twice as much the next. How had his weight changed?

h The skin diver went down 63 feet. After he came up 25 feet, he saw a beautiful piece of coral. How far was the coral beneath the surface?

i Mr. Swanson found he had gone 5 miles in the wrong direction. "Now I'm 32 miles from home." How far would he have had to go if he had not made a mistake?

j Bill walked 175 feet into a cave. Then he turned around and walked 210 feet. Where was he?

k The water had cooled 75 degrees Fahrenheit after it had boiled. This happened at the seashore last summer.

"Directed" number is one of the names that has been given to these numbers that can be used to label points on the Number Line on both sides of 0. Where do you think that name came from?

Others call them "signed" numbers. Why might that be an appropriate name?

Most mathematicians do not use arrows such as we have used ... $\overrightarrow{3}$ or $\overleftarrow{4}$: They prefer to indicate a direction from 0 in the following way:

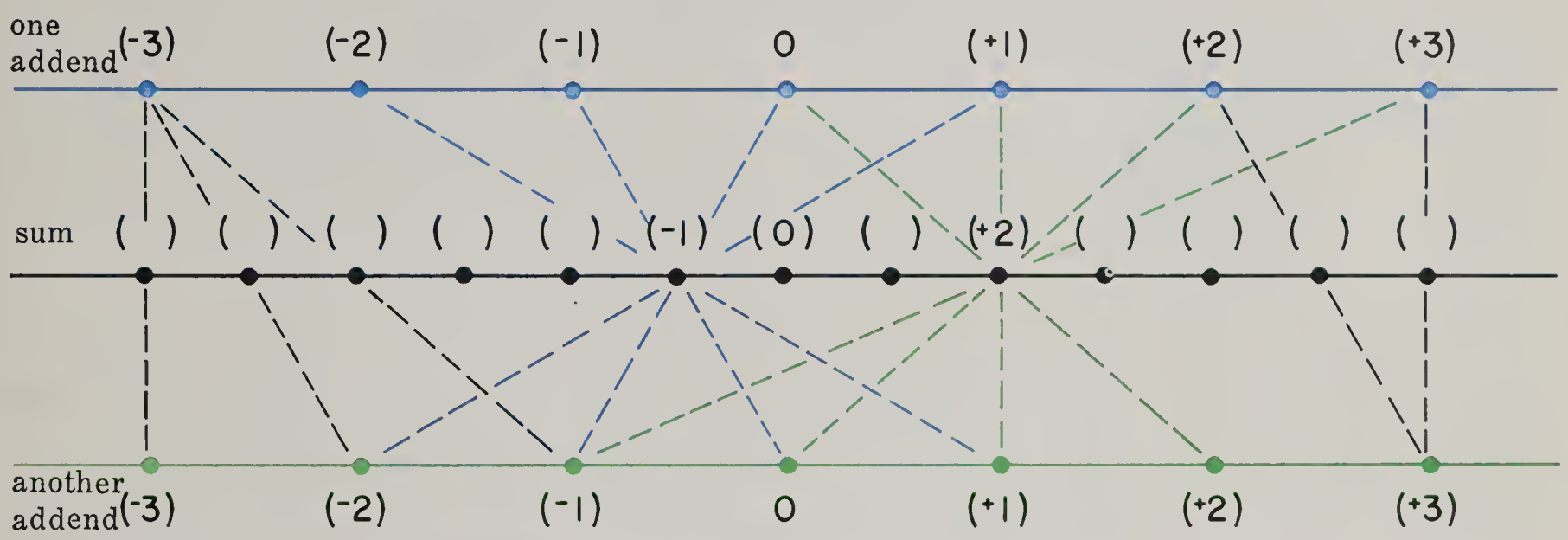
(+3) . . . read "positive three," and

(-4) . . . read "negative four."

We will begin using these preferred signs.

What do you suppose we can mean by ADDITION of Directed Numbers?

Here is a picture using the idea of the Number Ladder on its side:

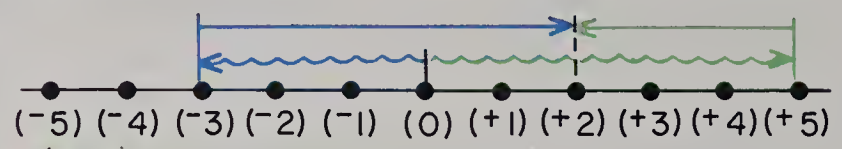


Can you finish labeling the middle line?

(+1) + (+1) = ( )	(+1) = (-1) + ( )	(+3) + (-2) = ( ) + (+3)
(0) + (+2) = ( )	(-2) = ( ) + (+1)	(-2) + (-3) + (+4) = ( )
(+2) + (0) = ( )	(-3) + ( ) = (0)	(+4) + (-2) + (-3) = ( )
(+3) + (-1) = ( )	( ) = (-1) + (+1)	(+2) + ( ) = (-3) + (-2)
(-2) + (+1) = ( )	( ) + (+2) = (+5)	(+9) + ( ) = (-5) + (+9)
(-1) + (0) = ( )	(-2) + ( ) = (+2)	(-4) + (-4) + (-4) = ( )
(0) + (-1) = ( )	(-5) = (-1) + ( )	(+5) + (+5) + (+5) = ( )
(+1) + (-2) = ( )	(-15) = (-7) + ( )	(-17) + (-8) + (+17) = ( )
(-3) + (-1) = ( )	(-6) + (+15) = ( )	(-25) + (+9) + (+16) = ( )
(-3) + (-2) = ( )	(+6) = ( ) + (-9)	(+) = (-) + (+) + (-)
(-3) + (-3) = ( )	( ) + (+3) = (-7)	(-) + (+) + (-) = (-)

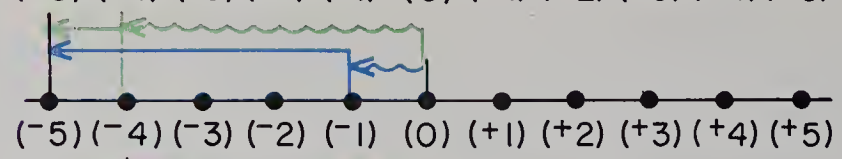
Another Picture to Suggest ADDITION of Directed Numbers.

$(-3) + (+5) = (\quad)$



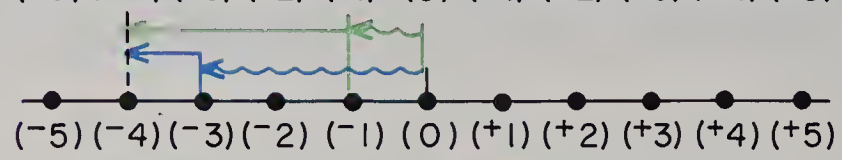
$(+5) + (-3) = (\quad)$

$(-1) + (-4) = (\quad)$



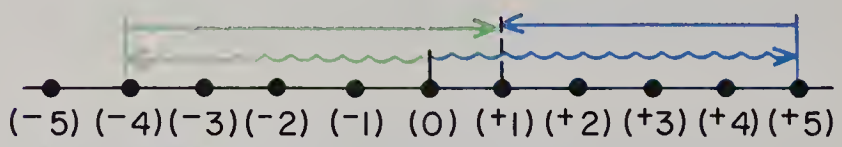
$(-4) + (-1) = (\quad)$

$(-3) + (-1) = (\quad)$



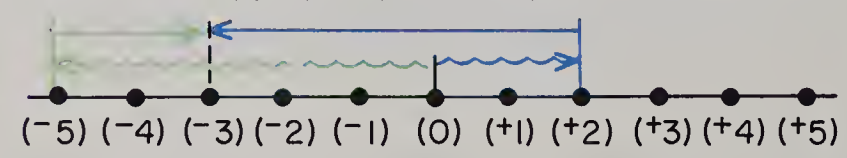
$(-1) + (-3) = (\quad)$

$(+5) + (-4) = (\quad)$



$(-4) + (+5) = (\quad)$

$(+2) + (-5) = (\quad)$



$(-5) + (+2) = (\quad)$

Rules for Cross Number Puzzles: Add across the top two rows; then add those results in the right-hand column. Next, add the numbers in each column; and add the results in the bottom row.

(+4)	(+3)	
(+2)	(+1)	

(−3)	(+5)	(+2)
(−1)	(−4)	

(−3)	(−4)	
(−2)	(−1)	

(+3)	(−5)	
(+1)	(+4)	

(+4)	(−3)	
(−2)	(+1)	

(−8)		(−4)
	(−7)	
		(−3)

(+15)	(+18)	
(−19)	(+8)	

(−7)	(+2)	
(+9)		
		0

(−4)		(−1)
(+2)		(+3)

(−2)	(+6)	
(−7)	(+2)	

	(+9)	
(−3)		
(−9)	(+6)	

	(−6)	

In a MAGIC SQUARE, the sum of the numbers in each row, column and diagonal are the same.

(+15)			
	(+6)		(+2)
	(+1)	(+5)	
		(+3)	
	(+15)	(+15)	

To the numbers in each cell of the Magic Square on the left,

ADD (−5) and

write the result in the respective cell on the right.

(+1)		

Example: (+6) + (−5) = (+1)

You may be surprised at the result above.

Add each pair of similarly located numbers in the Magic Squares above.

Examples:

(+6) + (+1) = (+7)

(+1) + (−4) = (−3)

(+8) + (+3) = ____

(+7) + (+2) = ____

(+5) + (0) = ____

(+3) + (−2) = ____

Is the result a Magic Square?

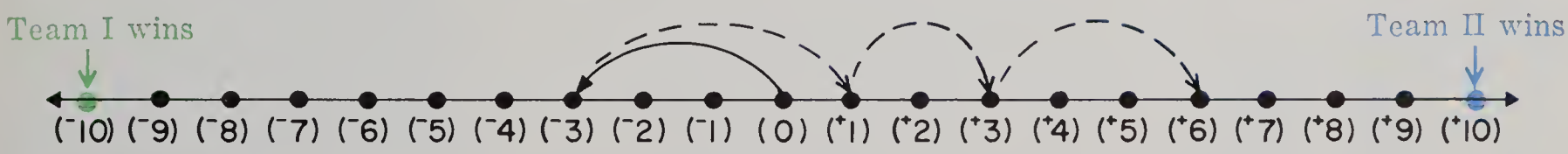
	(+7)		
	(−3)		

	A	B	C
X	(+1)	(+2)	(−3)
Y	(−4)	(0)	(+4)
Z	(+3)	(−2)	(−1)

Here is A Game by Sue Marson that the whole Arithmetic Class can play.

Team I has these cards: → X Y Z

Team II has these cards: → A B C



Each of two teams has a set of 3 cards — X, Y and Z or A, B, C. Each team selects one and places it on the table face down. Both are turned up — such as X by Team I and C by the other team.

Look at the Magic Square above. What number is in row X and column C? The answer (−3). Score is kept on the Number Line above as shown.

Teams select another card (or the same one) — such as C and Y which indicates a score of (+4). This is noted on the Number Line scoreboard.

If the next pair of cards were B and X, that moves the score (+2) — or, 2 to the right in this case. If AZ is next, the score moves 3 more to the right. If CY were next — (+4) — that calls for a jump to the right, landing at (+10), and Team II would win.

Complete the chart in each example. Then select one of the facts in the summary as a headline for a story.

In the first example you might select the fact that $20\% \times 50 = 10$ or 30% of $500 = 150$.

X	5	10	50	250
10%				
20%			10	
3%				

X	10	17	27	44
6				
9	90			
15				

+	.7	1.15	9.82
.7			
.69			
3.58	4.28		
10.25			

—	$\frac{3}{4}$	$1\frac{3}{8}$	$\frac{9}{16}$	2
$\frac{1}{2}$				$1\frac{1}{2}$
$\frac{1}{4}$				
$\frac{1}{8}$				

Ways	1	2	3	4	5	6	7	8
Mary								
Jane								
John								
Bill								
Total	13							

You Write the Headline.

One day, the Bowman children counted the books they brought home from the library. There were 13 in all.

"I brought 1 more than Jane," Mary said.

"John brought home the same number that I did," remarked Bill.

There are only 7 ways in which the numbers of books can be distributed so that the above statements are true.

I am thinking of a fraction. The denominator is not less than 4 nor more than 8. The numerator is not more than 40 nor less than 20.

Write fractions I might be thinking of. Also indicate the largest and the smallest fraction I might be thinking about — in their reduced form.

Largest Smallest

___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

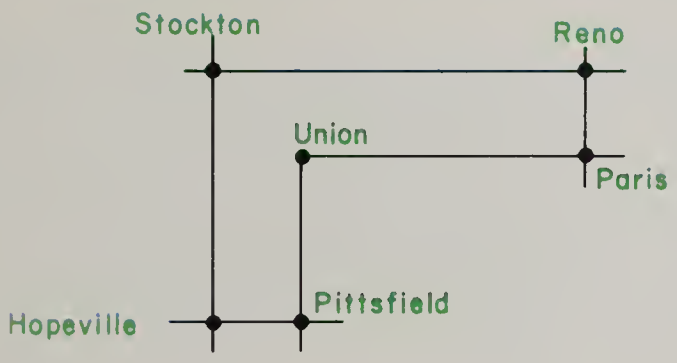
How many dollars in each possible purchase:		No. of cards @ 10¢			
		5	10	15	20
No. of cards @ 15¢	5				
	10				
	15				
	20				
	25				

Some greeting cards are sold in packages of 5 cards in each package. All cards in some packages cost 10¢ each. All cards in other packages cost 15¢ each.

Mrs. Baldwin spent $3.00 for some of each kind. How many 10¢ cards did she buy? _____

Could anyone spend $4.20? _____

Mrs. Jones spent $4.00. What can you say about her purchase?



Here is a map of some roads around Mr. Conrad's home town of Stockton. He lives 6 miles from Reno and 4 miles from Hopeville. It is $1\frac{1}{3}$ miles between Reno and Paris; and it is $1\frac{1}{2}$ miles from Hopeville to Pittsfield.

One day, Mr. Conrad drove on a route which took him through all these towns and back home. What is the shortest distance he could have driven?

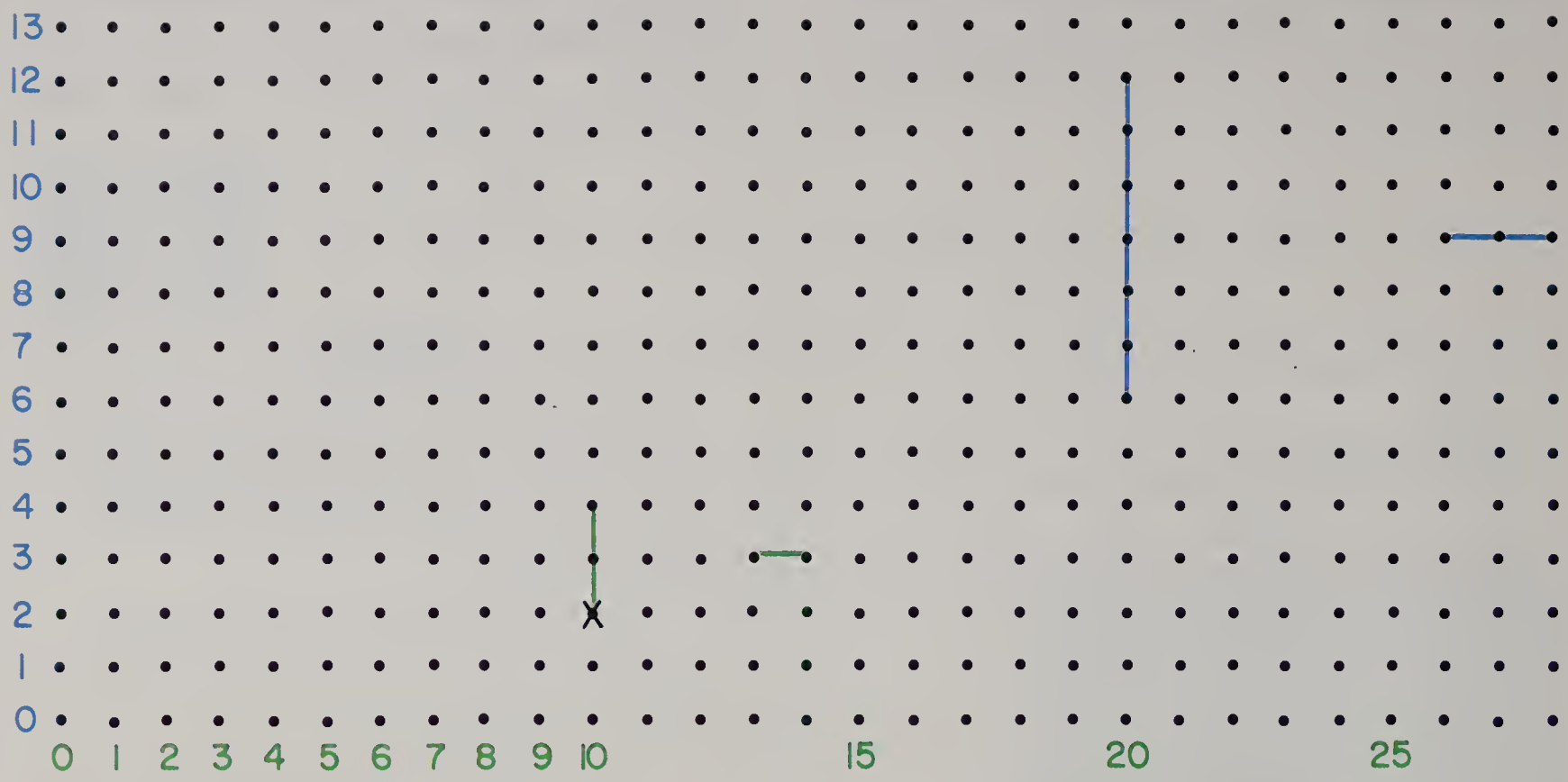
If the 1st day of the month fell on →	S	M	T	W	T	F	S
Then the second Monday would be →				13			
The following Wednesday would be →				15			

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

The band concert is always held on the first Wednesday after the second Monday. On what days of the month might it occur?

From the information given, indicate the cities from west to east.

Dodge City is west of Badger, but east of Malta. Justice is west of Benton, but east of Badger. Danville is between Malta and Dodge City.



Each pair of numbers indicated in each group below locates a particular point or dot.

Start at 0; move to the right as far as the first member of the pair indicates; then move up as far as the second member of the pair indicates. Mark your landing point.

In the first example, the pair — 10, 2 — leads to the point marked with a cross.

After locating the first dot in each group, draw a straight line to connect each newly located dot with the one found just before it.

When the first part of each group, labeled with Roman Numerals, is completed, repeat the first part of the example after multiplying the first member of each pair by 2 and the second member by 3.

I a

→	↑
10	2
10	3
10	4

I b

→	↑
20	6
20	9
20	12

III a

→	↑
3	4
3	2
2	2
2	4

III b

→	↑
6	12
6	6
4	

IV a

→	↑
6	3
7	3
7	4
6	4
6	2

IV b

→	↑
12	
	9
14	
	6

V a

→	↑
5	2
4	3
5	3
5	4
4	4
4	2

V b

→	↑
10	6

II a

→	↑
13	3
14	3

II b

→	↑
26	9
28	9

VI a

→	↑
14	2
13	2
13	4
14	4

VI b

→	↑
26	
	12

VII a

→	↑
12	2
11	3
12	4

VII b

→	↑
22	6
24	9
22	12

VIII a

→	↑
8	4
8	3

VIII b

→	↑
16	6
18	12
18	9
18	6

IX a

→	↑
0	3
1	3

IX b

→	↑
2	12
0	12
2	6
0	6

NUMA GOES EXPLORING.

Numa liked to explore the many paths in the park. The sketch on this page shows some of the paths.

On her first trip, she started out on the black path until it branched to the North and to the South. She decided to go South. When the path branched a second time, she went South again. At the third branch, she made the same decision. But at the fourth branch, she turned North and she came to a spot marked with a letter. What letter?

She always knew her way back. She simply went West whenever she could.

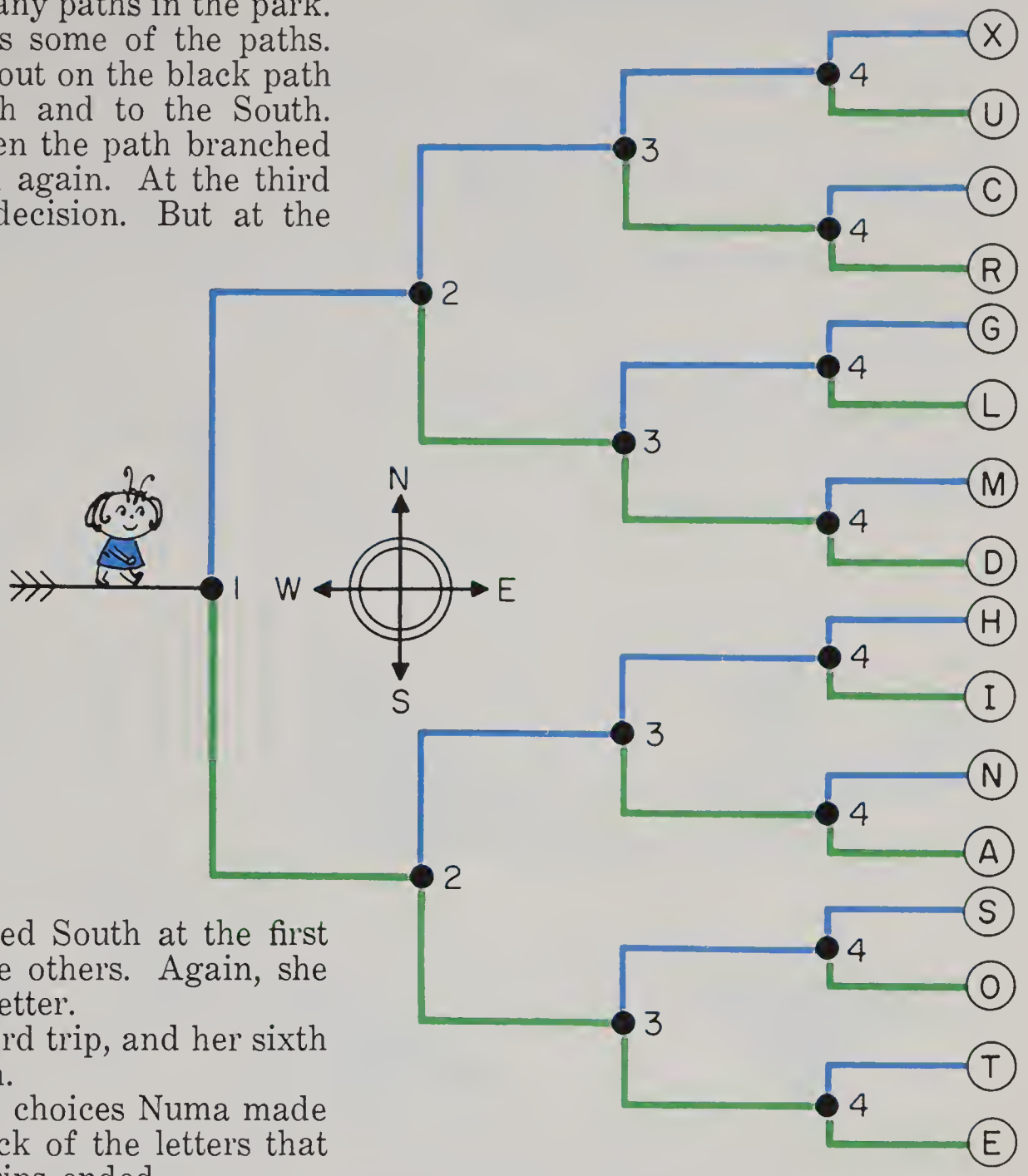
On her second trip, she turned South at the first branch and then went North whenever she had a choice until she came to a letter. What letter?

On her third trip, she turned South at the first branch, then North at the second and third branches, and South at the last branch. This brought her to a spot marked with a letter. What letter?

On her fourth trip, she turned South at the first two branches and North at the others. Again, she reached a spot marked by a letter.

Her fifth trip was like the third trip, and her sixth trip was the same as the fourth.

Fill in the chart showing the choices Numa made in her first six trips. Keep track of the letters that marked the spots where her trips ended.



Find where she went on trips seven through 17.

Choices	Trips Numa took																
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th	14th	15th	16th	17th
at 1	S						S	S	S	N	N	S	S	N	S	N	S
at 2	S						N	S	S	N	N	S	S	N	S	S	S
at 3	S						S	N	S	S	S	S	S	S	N	S	S
at 4	N						S	N	S	N	S	S	N	N	S	S	S

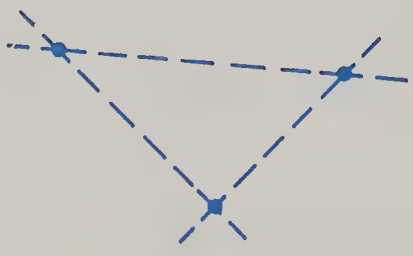
Find the secret message below. G stands for Green and B stands for Blue.

Choices																											
at 1	G	B	B	B	G	B	B	G	G	G	B	B	B	B	G	B	G	B	B	B	G	B	B	G	G	B	B
at 2	B	B	G	B	G	B	B	B	B	G	B	B	G	B	G	B	G	B	G	B	G	B	B	G	G	B	B
at 3	B	G	B	B	G	B	B	G	G	G	G	B	G	B	B	G	G	B	G	B	G	G	B	B	B	B	B
at 4	B	B	G	G	G	G	B	B	B	G	B	G	G	B	B	G	G	G	G	B	G	G	B	G	B	B	G

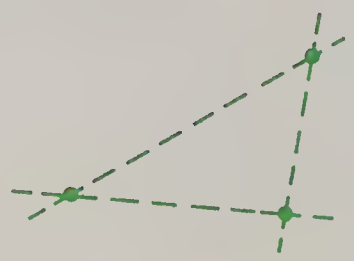
Surprises

"Here's a trick," Mary explained to Bill, "that has an interesting surprise to it. Mark three points on your piece of paper and draw a line through each pair of points. Then cut along those lines and a piece will drop out.

"Sure," Bill said, "and the pieces that drop out will be triangles."



Bill's triangle



Mary's triangle

"Now color each corner, and then tear the triangle into three parts, a corner to each part.

"Next," Mary continued, "I will paste one of my pieces with the marked point touching this dot I am drawing. Then you paste one of your pieces with its point on the dot — and right next to mine.

A corner of Mary's triangle

A corner of Bill's triangle

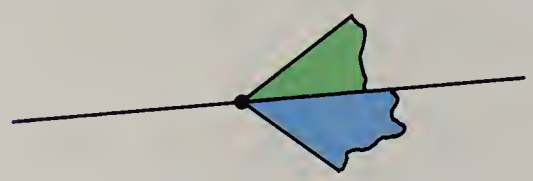
"Then I will paste another corner of my triangle next to yours; you do the same; and I will paste my last corner down. And you paste your last corner down."

"It fits!" Bill exclaimed. "Will wonders never cease!"

"Well," Mary said with a twinkle in her eye, "here is another surprise for you. Start out just as we did before. Draw any shape triangle you like, and so will I. Mark its corners, and tear it so each piece contains one corner.

"This time, I'll mark a point on a line and paste one of my corners on one side of the line and you do the same on the other side."

Here is the result:



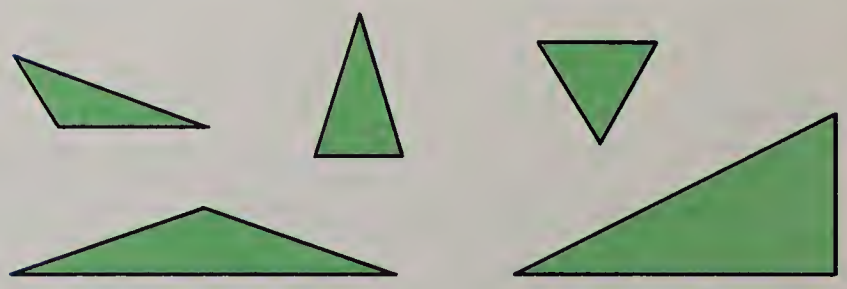
"Then I'll paste a second corner next to my first and you do the same on your side."

"Finally, I'll paste my last corner down and you do the same," Mary explained.

"Amazing!" Bill said. "Let's try it again, but this time I'm going to change the shape of my triangle."

What was the surprise?

Try it yourself. Cut out two triangles of any shape and repeat both of the experiments. Here are some shapes to consider:

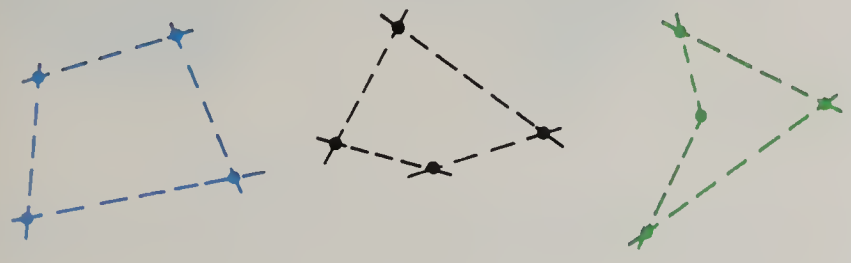


Is there anything particular to talk about in each of these triangles?

Bill's Surprises

The next day, Bill came to class with a surprise of his own.

"Mary," he said, "I worked out a trick something like yours. Mark any four points and connect them so you have a four-sided figure. Like this:



"Now, cut along the line that connects the points. Next, tear each shape into four pieces so that each contains one of the corners — one of the original four points.

"Then, arrange each group of four pieces so that the corners touch at the points, and so that their sides just touch.

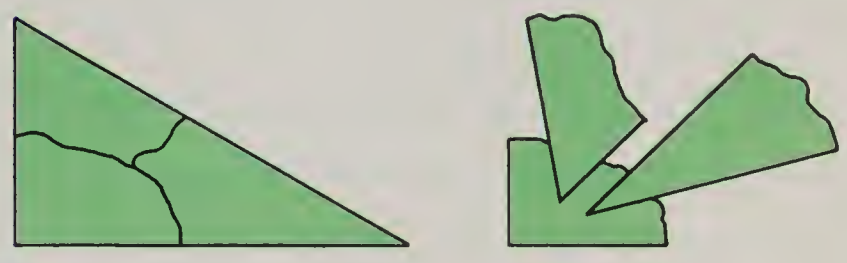
"Surprised?" Bill asked.

"I surely am surprised," Mary replied. "Would it work for squares, rectangles, and other four-sided shapes?"

"Try it!"

"The other fitting trick is this: Cut the corners of a rectangular piece of paper so that the corner cut-offs are different shapes and sizes.

"Each triangle corner cut-off has a square corner. Tear each triangle in three pieces so that one of the corners is in each piece. Try to fit the other corners of each into the square corner.



"Try this experiment in 'fitting' with each of the triangular pieces that has a square corner."

"It's a 'fitting' surprise," Mary agreed.

Tom has been watching and trying some examples of his own.

"I think all the results would be true," he said, "even if you turned all the pieces over or just some of them."

"That's hard to believe," Mary said.

"Try it, then," said Tom.

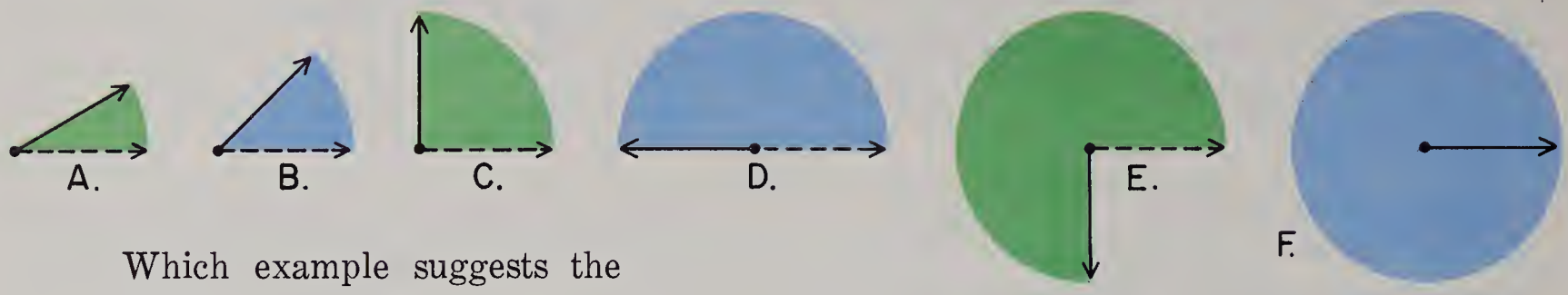
Make a study of your own of these fitting tricks. Try each trick with many different shapes. Turn some of the pieces over.

As you try each experiment, paste down your results so they become a record of these examples of 'fitting.'

Perhaps you can invent a 'fitting trick' of your own.

Swinging Around a Pivot.

Imagine a pointer or clock-hand that can be swung around a pivot. Before it has been swung, suppose it points to the right as shown by the broken line. Of the many positions into which it can turn, a few are shown.



Which example suggests the following amounts of turning?

— smallest amount of turning ____
— half-way around ____
— one-fourth of the way around ____
— all the way around ____
— $\frac{3}{4}$ of the way around ____
— $\frac{1}{2}$ of a quarter of a full turn ____

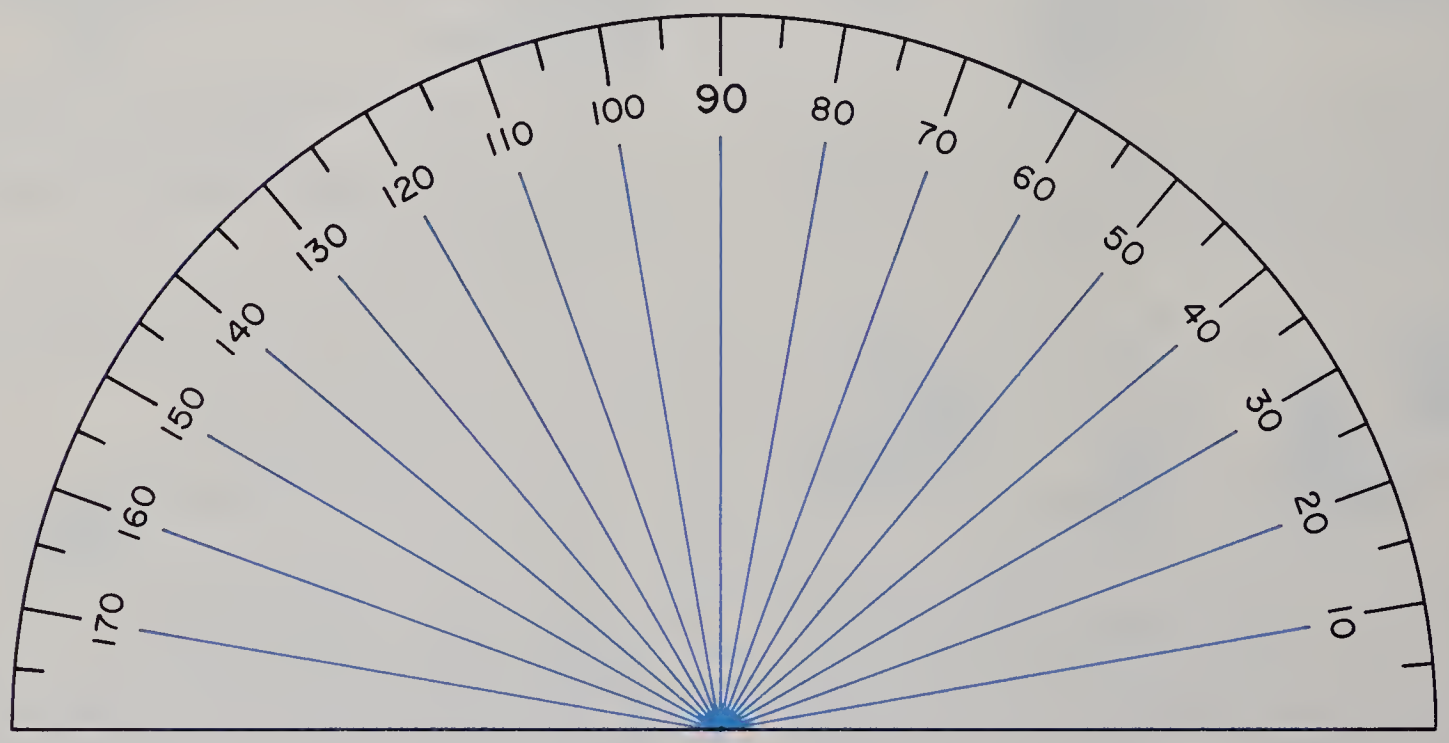
Mathematicians have agreed to consider a quarter of a full turn as a turn of 90 degrees — in shorthand, 90°.

Then, a half of a turn is ____°

Three fourths of a turn is ____°

A full turn must be ____°

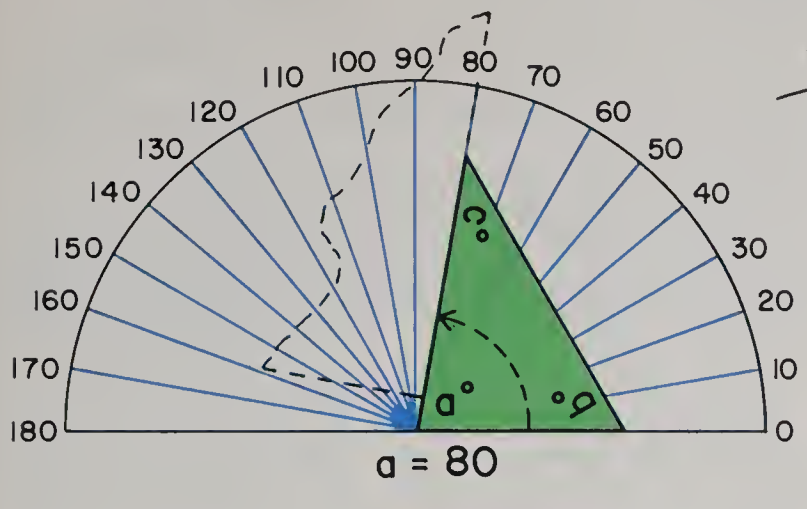
An instrument such as the one below is called a PROTRACTOR. It is used to measure the amount of turning that would be required in making a particular angle.

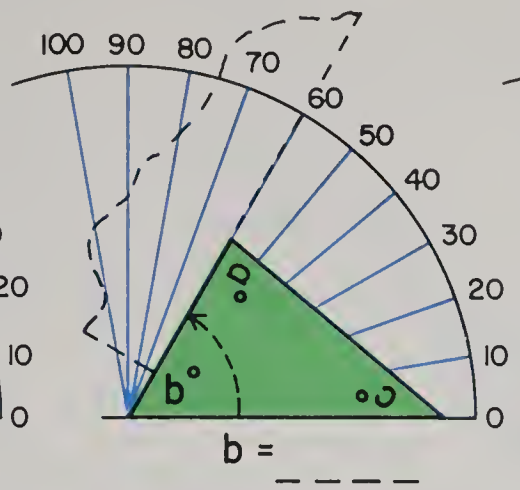


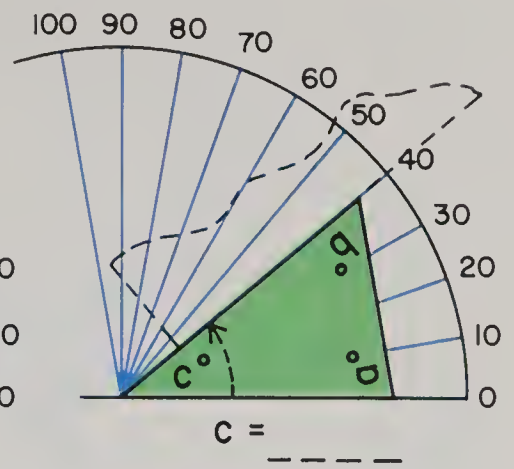
The protractor is a most valuable tool. It is used to measure the size of angles — the amounts of turning it would take for a hand to sweep from one side of a triangle to another — the angle included between these sides.

Cut out triangles of many different shapes and sizes. Mark the angle of each — a, b, and c — on the protractor. Measure each angle. Record the results in the chart on page 103. What do you notice about the sums of each group of angles?

If triangles are small and do not cut the scale, use a piece of paper along an edge, as the dotted black line suggests, to extend their sides for measuring.



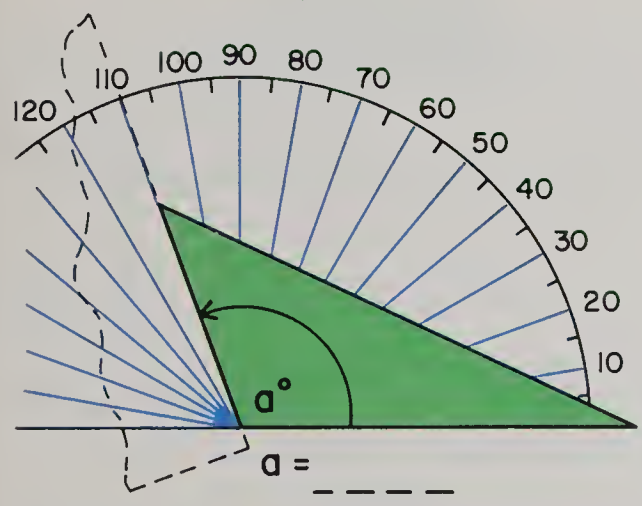


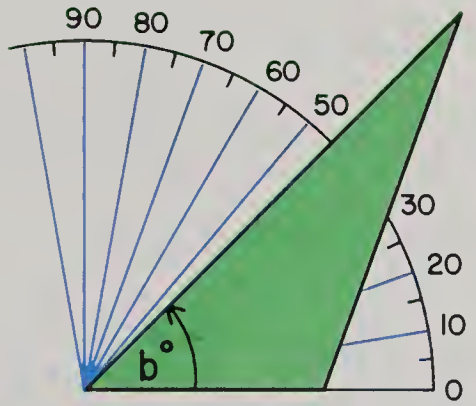


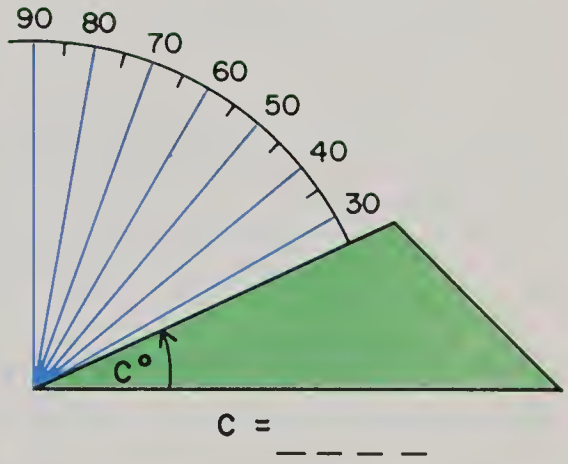
Record your experiments with this and other triangles.

Examples:

	Above	Below	F	G	H	I	J	K	L	M	N	O	P
a	80												
b													
c													
Total													



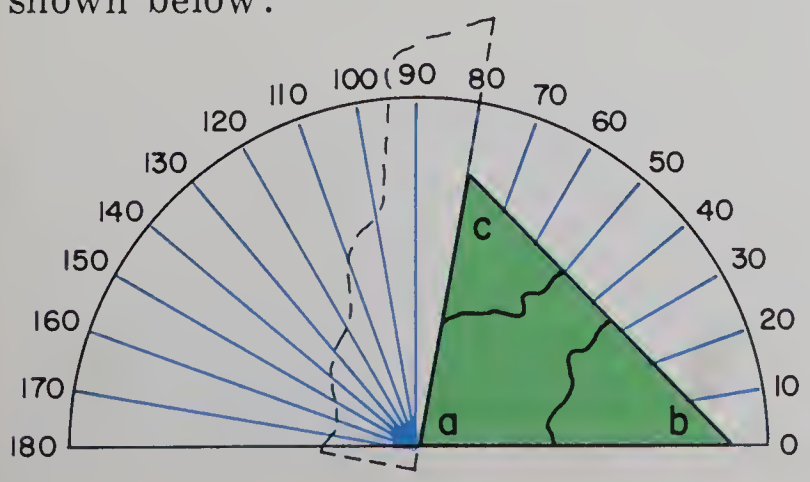




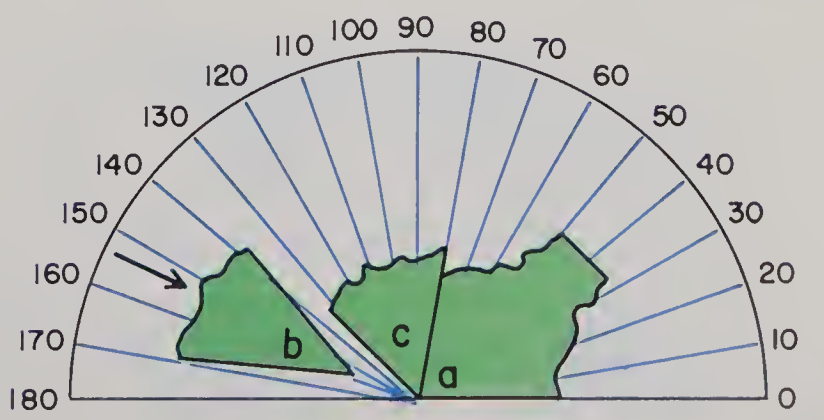
For your other experiments, cut triangles out of paper and cardboard. Place them on the protractor on page 102, measuring each angle in turn and recording results.

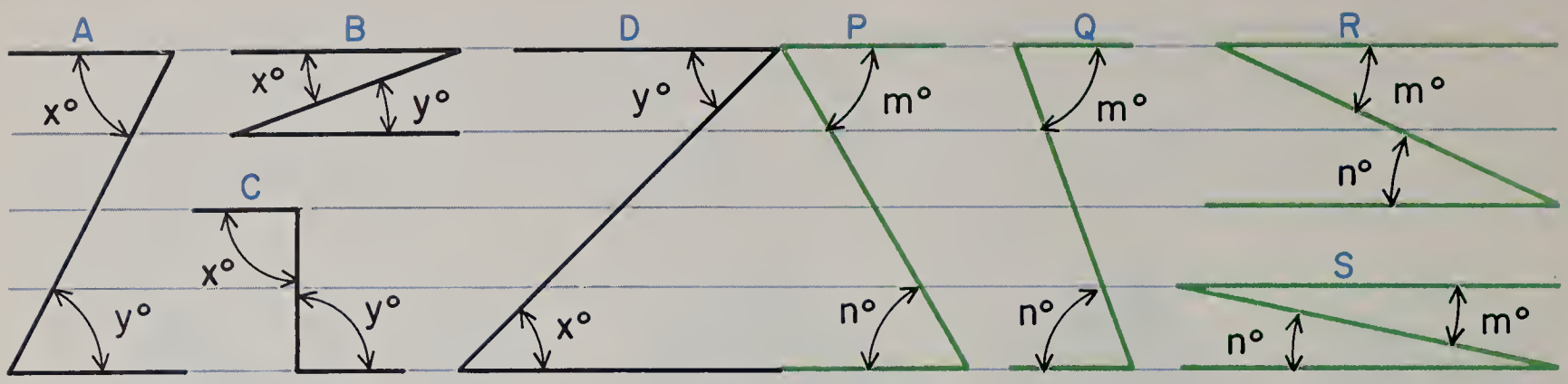
If your measurements were exact, what would you expect about the sum of the number of degrees in each group of three angles.

Use a triangle of any shape. Paste one corner down as if you were measuring it, as shown below:



Tear off other corners. Fit one as shown. What will happen if the other corner is pushed into place?





In the Z-shapes above, use a protractor to measure angles marked x° and y°. In the backwards Ƨ-shapes, measure angles marked m° and n°.

On lined paper, draw some Z's of your own — of the front and backward varieties — and record similar measurements of the angles on the chart below.

Number of degrees	A	B	C	D	E	F	G	H	I	J	K
x degrees											
y degrees											

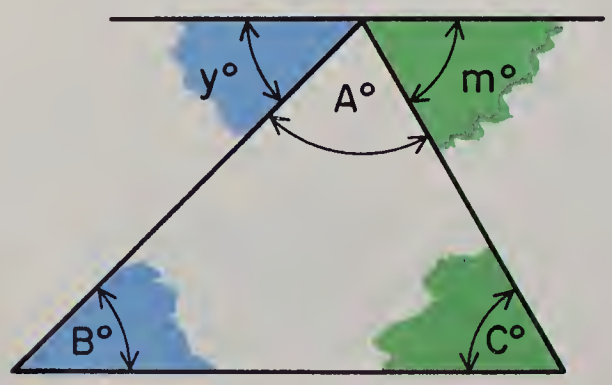
Number of degrees	P	Q	R	S	T	U	V	W	X	Y	Z
m degrees											
n degrees											

In shorthand, we can say very briefly what we notice: x = y and m = n. In the Z and the reverse Ƨ the angles are the same size.

We call this the "Z-result."

(Turn this page on the side and you might decide to call it the "Ɲ-result.")

Below is a copy of the shape or design that appears in the center at the top of the page:



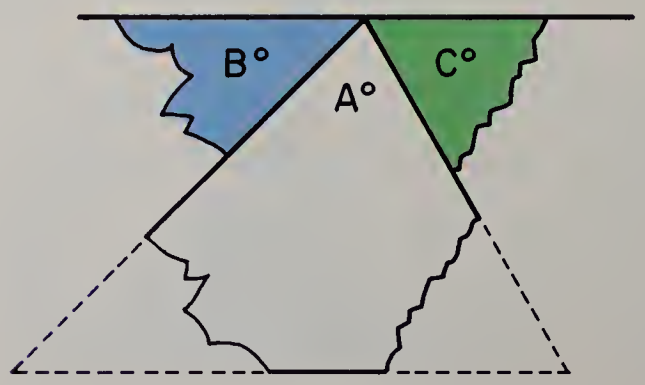
Only the labels have been changed, and a little color has been added.

You can see it as a Z and reverse Ƨ put together — or as a triangle with a line through one corner.

One Z-result tells us that B = y.

Another Z-result tells us that C = m.

If we cut out the corner marked B° and C° and respectively pasted them on angles marked y° and m°, we would have:



Could you neatly tile a floor using triangular tiles all the same size and shape? We will assume that you can cement pieces down with either side up. Also, you do not need to worry about the corners or the edges of the room.

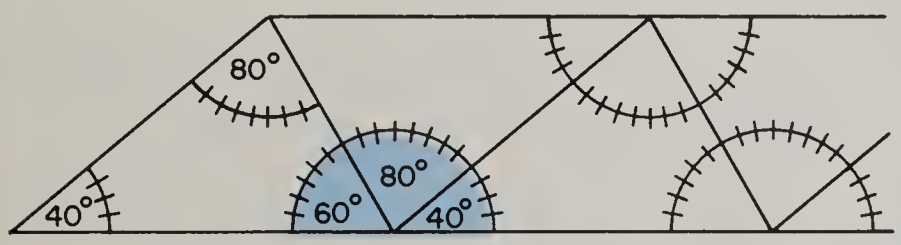
Draw a triangle on the top sheet of several sheets of paper. Cut as many at a time as possible — so you will have a good supply. (It's more fun if the angles of your pattern triangle are all different sizes.)

Here is one example:

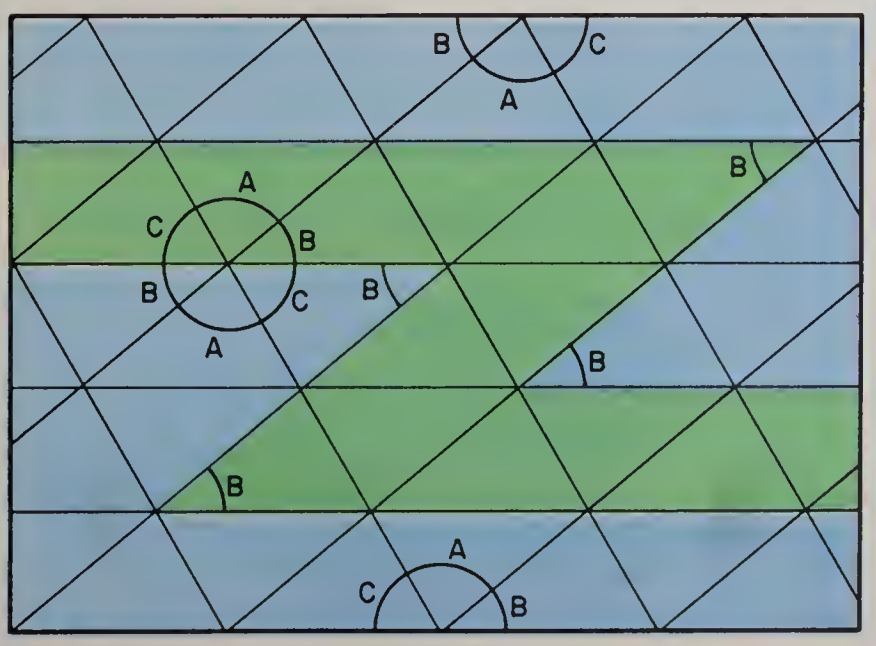
A = 80° B = 40° C = ____

[Can you tell without using a protractor?]

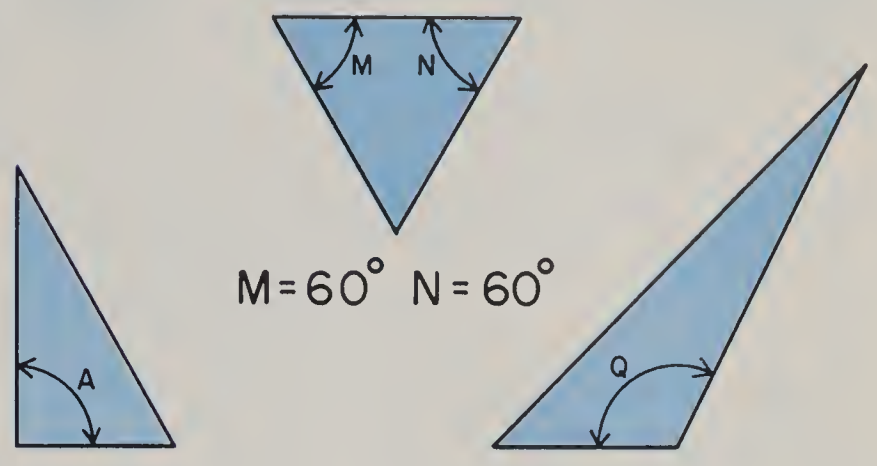
Here is a bridge design:



A pattern for a floor of tile the same shape (but smaller) will look like this, some green and some blue.



Make some floor covering designs of your own. Each time you finish a design, try a different shape. Here are some you might like to choose from:



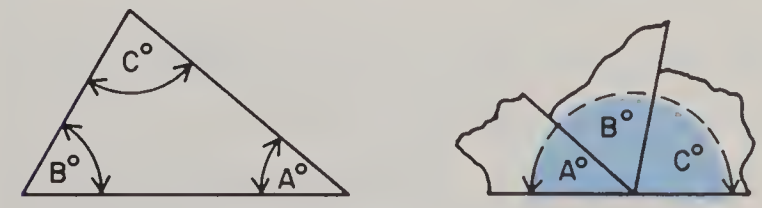
(Cut out chunks of the patterns you make and paste them in your book on this page.)

You know that a quarter of a circle contains 90 degrees. So, one degree is found by dividing a quarter of a circle into 90 parts — each of which is one degree.

If a quarter of a circle contains 90 degrees, then a half of a circle must contain 90 + 90 or 2 × 90 or 180 degrees.

The experiments in these pages have probably led you to the conclusion that if you tear the three corners off any triangle, and fit them together in a certain way, they will neatly fill half a circle.

That is a conclusion you can rely upon, and it is a most useful fact to understand.

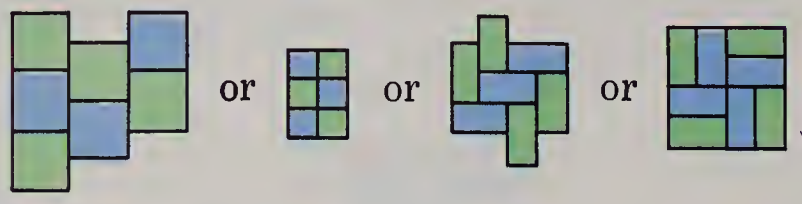


In shorthand, we write: "If a triangle has angles of sizes A°, B° and C°, then:

$$A° + B° + C° = 180°$$

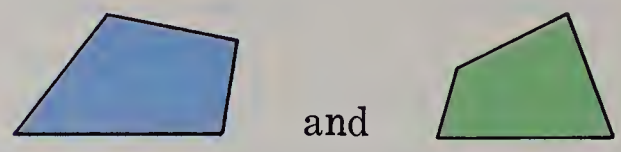
Experiments with Four-sided Tiles.

Of course you can work out many patterns in tile with a square or with a rectangle — and these are four-sided shapes.



. . . . and many others!

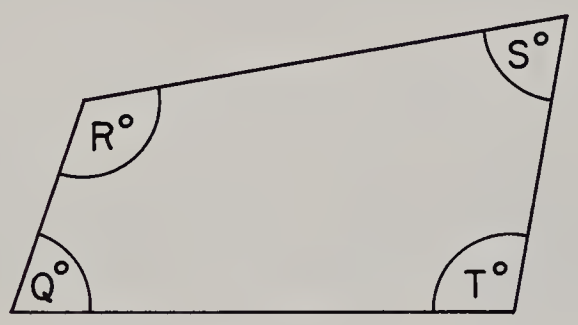
It's harder when all sides are different in length. Try a shape like this:



Start with a large supply of the shape you select. Could you cover a floor with such Tiles?

NOTE: Try your experiment before going on. If you need help later on, you can find hints below.

In experiments, we found that the angles of a triangle add up to 180°. In a four-sided figure, there are four angles. In the example below, what do you think Q° + R° + S° + T° will be?



Measure the angles of several four-sided figures before you go on. Keep your record below.

No. of degrees	Above	Your experiments			
in Q					
in R					
in S					
in T					
Total					

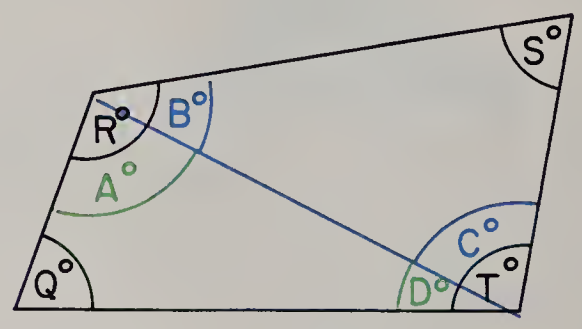
You may want to include a square and a rectangle in your experiments . . .

Can you finish this shorthand statement: "If a four-sided figure has angles of sizes Q°, R°, S° and T°, then:

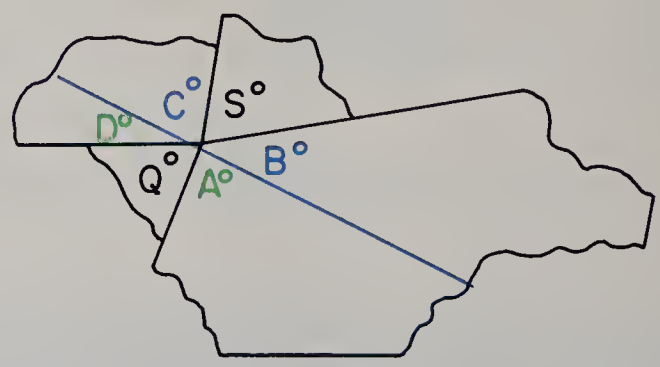
Q + R + S + T = ____

You can reach the same answer by reasoning — without any measuring.

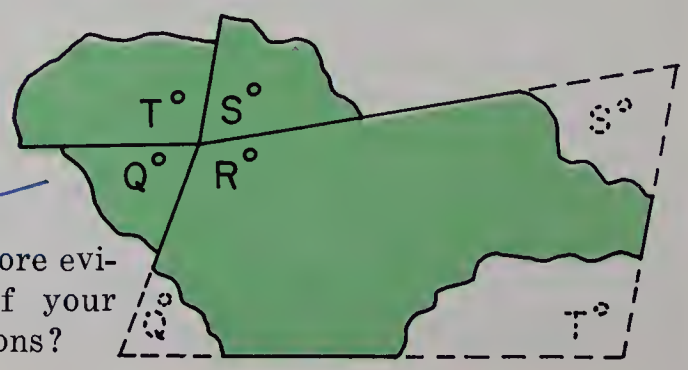
Draw a line that breaks the figure into two triangles — does this give you any ideas?



If we tore out three corners and fitted them with the other, we might have:



You can tile the floor if you study the diagram below:



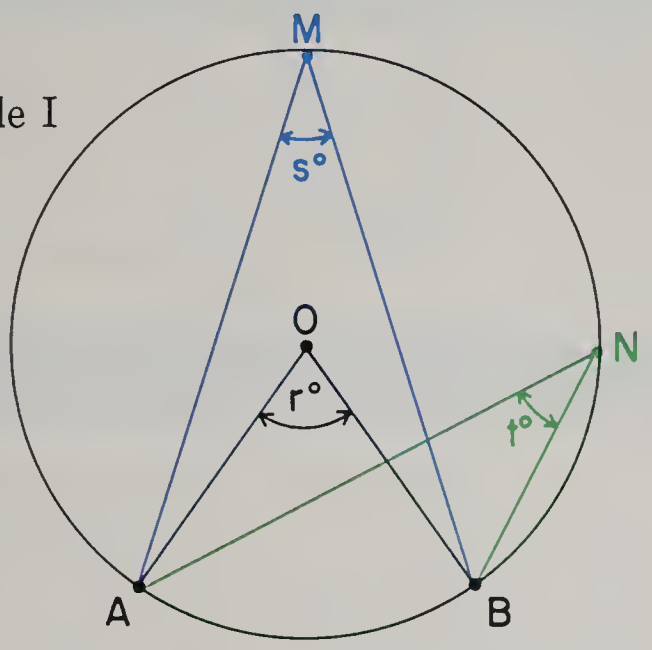
Is this more evidence of your conclusions?

(Study this trick on your own or with another student. Find out how much you can learn through your own experiments.)

"I learned a nice trick from my brother," Betty announced. "He is studying geometry. The trick goes like this:

"Draw a circle any size and number the center with the letter 'O.' I want Harry to choose two points on the circle and label them A and B. Connect A to O and B to O.

Example I



"Mark the angle formed as r°.

"Now I want Ruth to find another point on the circle and label it M.

"I want Bob to find another point on the circle and label it N.

"Draw lines from A and B to M. Label the angle formed as s°. Draw lines from A and B to N. Label the angle formed as t°.

"Now," Betty reminded them, "I had Harry and Ruth and Bob do the drawing.

"I want Harry to use a protractor and measure the angle marked r°." Harry did. "It's about 70°," he said.

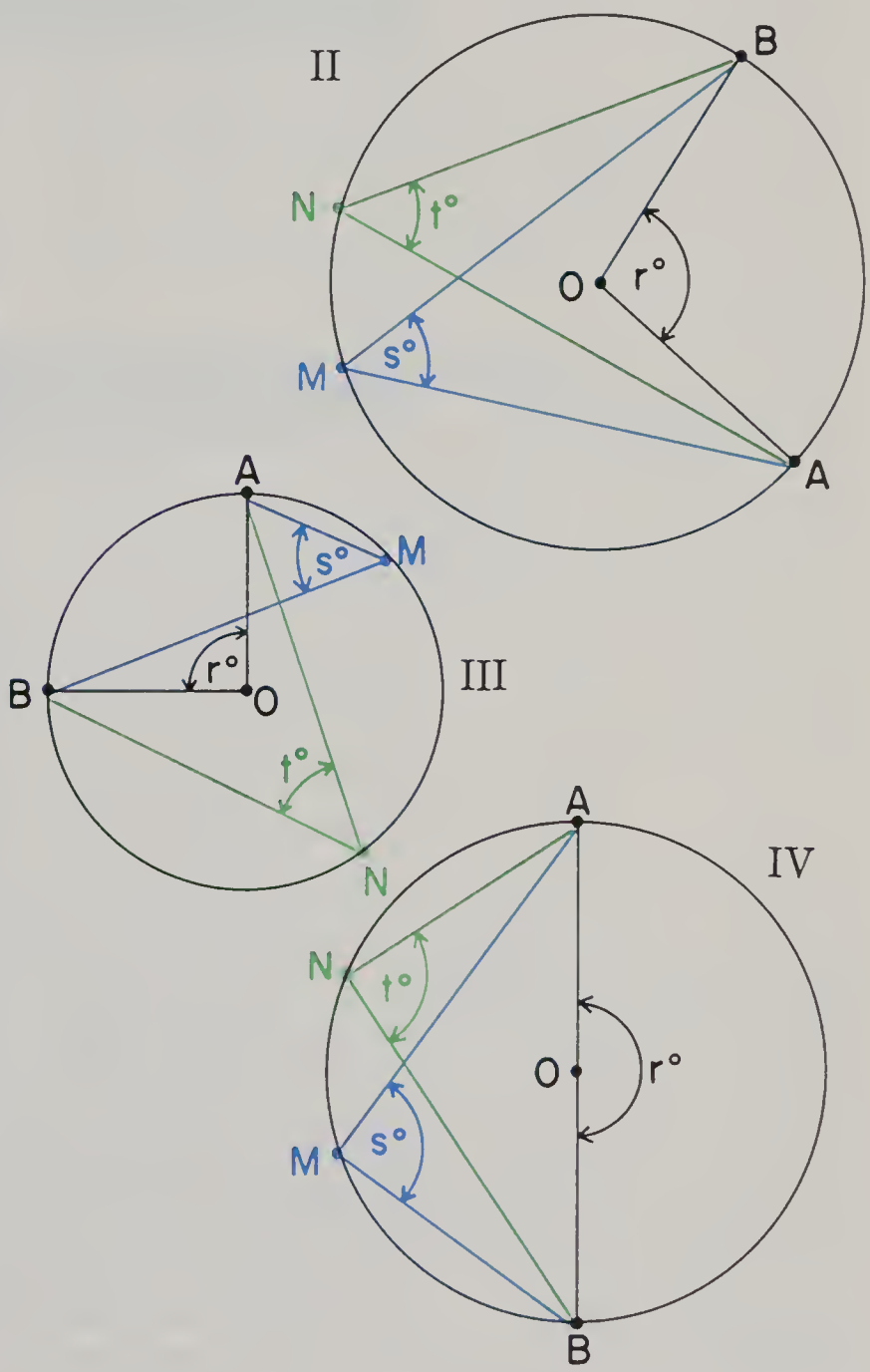
"Now Ruth can measure the angle marked s° and Bob can measure the angle marked t°. Unless Ruth or Harry or Bob make a mistake, Ruth will find that s = 35 and Bob will find that t = 35."

Bob and Ruth took turns with the protractor.

"Betty's right," Bob exclaimed.

"She certainly is," agreed Ruth.

"Let's try it again," Harry urged. They did. Here are some of their results. Record what you find below.



Number of Degrees

	r	s	t
I	70	35	
II			
III			
IV			

Think of a Number.

"I once played a game that went like this," Harry said.

"Think of a number. Don't tell me, but write it down, so you won't forget.

"Add 3. Double your result. Take away 4. Divide by 2. Take away the number you thought of first. If your answer isn't 1, then you've made a mistake!"

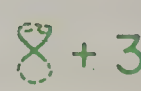
Most of the class was amazed — except one or two who remembered the trick. The class made a chart of what had happened.

(Complete the record.)

	Al	Bob	Don	Meg	Art	Sam	Alice	Jim	Nancy	Paul	Helen
Think of a number	5	7	0	3	25					158	589
Add 3	8					34					
Double the result	16						44				
Take away 4	12							56			
Divide by 2	6								106		
Take away the number you thought of first	1										

The picture below suggests the way this trick works

A convenient shorthand

Think of a number	THINK OF A NUMBER! Imagine you put that many counters in a bag. Don't tell anyone how many.	
Add 3	ADD THREE. Put 3 counters by the bag. Now you have a bag and 3 counters more.	
Double it	DOUBLE IT. You need another bag like the first — and another 3 counters. That's 2 bags and 6 counters.	2x + 6
Take away 4	TAKE AWAY FOUR. That leaves 2 bags and 2 counters.	2x + 2
Divide by 2	DIVIDE BY TWO. Take away one bag and one counter, leaving one bag and one counter.	x + 1
Take away no. thought of first	TAKE AWAY THE NUMBER YOU THOUGHT OF FIRST. No matter how many counters you had in it. One left!	1

Remarks about the shorthand.

In the shorthand, we erase the top and bottom of the bag — and the remainder looks much like the letter "x". Of course, any letter would do to stand for "the number you thought of first" — "n" or "y" — you choose it.

Instead of drawing counters, we used numbers to keep track of "how many."

Instead of drawing two bags, we shortened that to "$2x$". In other games we may want to show 100 bags, and $100x$ is a short way of indicating that.

More about the Shorthand.

We will use "x" as shorthand for "any number you think of." We could use any mark or letter, but on this page we will use "x."

In the spaces that follow, choose your answer from the expressions in the blue-tinted block at the top of the next column.

Choose the expression that best fits the statement given. The first two blanks are filled in to show what is wanted.

x $\quad x+1 \quad x+2 \quad x+3 \quad 2x$

$x-1 \quad x-2 \quad 3x \quad x^2$

(x^2 is shorthand for x times x — "x squared.")

a. Any number you may think of x

b. One more than the number you may have thought of $x+1$

c. One less than the number you thought of _____

d. Two less than the number you thought of _____

e. Three more than the number you thought of _____

f. Three times the number you thought of _____

g. Two more than the number you thought of _____

h. The number you thought of multiplied by two _____

i. The number you thought of multiplied by itself _____

j. Twice the number you thought of _____

k. The number you thought of with one subtracted _____

l. The number you thought of with two added _____

m. Three added to the number you thought of _____

n. Three multiplied by the number you thought of _____

o. The square of the number you thought of _____

p. The number you thought of multiplied by one _____

"Think of another number," Harry broke in.

"Then follow these directions: First double it. Next, add 9. Then add the number you thought of first. Divide by 3. Double this result. Subtract 6 and tell me your answer. I'll tell you the number you thought of first."

	Shorthand	Al	Bob	Don	Meg	Art	Helen	Alice
Think of a number	x	4	2					
Double it	$2x$	8	4					
Add 9	$2x+9$	17						263
Add the number you thought of first	$3x+9$	21						
Divide by 3	$x+3$	7					78	
Double this result								
Subtract 6		8			0	100		
The number you thought of is	x	4	2	1				

More Shorthand.

Each series of expressions include some of the different ways of saying the same thing. The usually preferred form is printed on a green background.

a. Two times three . . . three times two . . . 2 multiplied by 3 . . 2×3 . . . $2 \cdot 3$ . . . $3 \cdot 2$.
(The multiplication sign looks so much like the letter 'x' we often use a dot in its place.)

b. The number you thought of times 4 . . . $x \cdot 4$ . . . $4 \cdot x$ . . . $4x$.
(Where no confusion will result, we omit the dot. Why can't we omit the dot in $2 \cdot 5$?)

c Twice the sum of five and nine . . . $2 \times (5+9)$ . . . $2 \cdot (5+9)$ . . . $2(5+9)$.
(We can omit the dot here because no confusion results.)

d. The number you thought of . . . x . . . a . . . n . . . y . . . z . . . b . . . etc.
(Any letter is as good as another. Be sure each time everyone understands what you mean.)

An agreement about order of work.

To prevent confusions, mathematicians agree they mean that all indicated multiplications and divisions should be carried out first. Only then should indicated addition and subtraction be carried out. In cases of exceptions, parentheses are placed around the part of the expression to be tackled first.

Without this agreement, we could not tell which of the following was indicated.

Which one?

$2 + 4 \cdot 3 = 6 \cdot 3 = 18$

$2 + 4 \cdot 3 = 2 + 12 = 14$

Certainly 18 and 14 are very different results. But, because of our agreement about order of work, we know that the indicated multiplication should be carried out first.

If it were intended that the addition should be done first, that can be indicated by use of parentheses. It would be:

$(2+4) \cdot 3$ or $(2+4)\ 3 = 6 \cdot 3 = 18$

With this agreement in mind, complete the mathematical sentences – a through l – in the column at the right.

a. $2 \cdot 3 + 1 =$ ____
b. $2 + 3 \cdot 4 =$ ____
c. $3 \cdot 2 - 1 =$ ____
d. $1 + 2 \cdot 3 =$ ____
e. $1 + 3 \cdot 2 =$ ____
f. $(2+3) \cdot 4 =$ ____
g. $2 \cdot (3 - 1) =$ ____
h. $4\,(3 + 2) =$ ____
i. $(3 - 1)\,5 =$ ____
j. $3\,(2 - 1) =$ ____
k. $(1 + 2)\,3 =$ ____
l. $(1 + 3)\,2 =$ ____

In the lines below, is the number printed in green equal to the number printed in blue?

Are they equal?

$1 + 2 \cdot 3$ ..... $1 + 3 \cdot 2$ ..... _____

$(1 + 2)\,3$ ..... $(1 + 3)\,2$ ..... _____

$2 \cdot 3 + 1$ ..... $2\,(3 + 1)$ ..... _____

$(2 + 3)\,4$ ..... $2 \cdot 4 + 3 \cdot 4$ ..... _____

$2\,(3 + 4)$ ..... $(3 + 4)\,2$ ..... _____

$3\,(2 + 4)$ ..... $3 \cdot 2 + 3 \cdot 4$ ..... _____

$2 + 3 \cdot 4 + 5$ ..... $(2 + 3) \cdot (4 + 5)$ .... _____

$3x + 2$ ..... $3\,(x + 2)$ ..... _____

$2\,(x + 7)$ ..... $2x + 14$ ..... _____

More Examples of Harry's "Think of a Number Trick:"

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
Think of a number	x	2						
Add 10	$x + 10$	12						
Triple it	$3x + 30$	36			93			35.4
Subtract 14		22				190		
Add no. thought of first		24					18	
Divide by 4		6	4	14				
Your number is	x	2						

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
Think of a number	a							$\frac{5}{8}$
Add 2	$a + 2$						$3\frac{1}{4}$	
Multiply by 6								
Subtract 12								
Divide by 3	$2a$							
Subtract 1		5	9	1	19	0		
Your number is								

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
Think of a number	n							1.07
Add 3 times that number							.4	
Add 6						106		
Divide by 2		23	3	17	33			
Your number is								

*Be careful! This one is different! Some of the directions are missing.

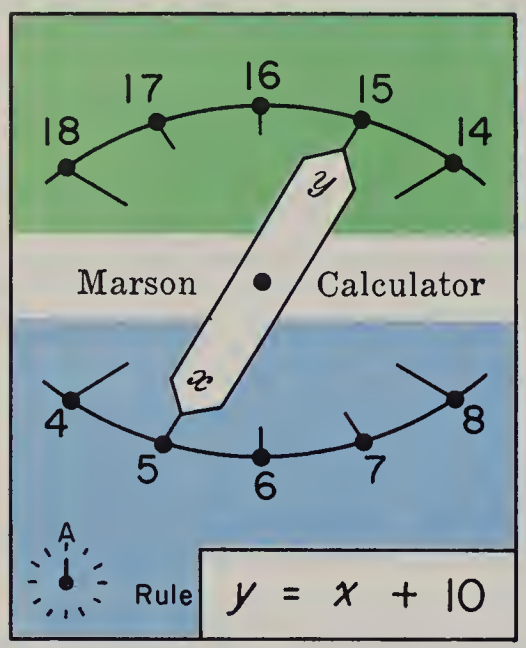
	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
Think of a number	y	10			.3			5
					3.3		$3\frac{1}{2}$	8
Multiply by 100			300			1,000		
								750
Divide by 25			10					
Subtract 6		44	4	8				
Your number is								

The Marson Computor.

One day, Mr. Wilson, the principal, brought in another of Alec's machines.

"It's a calculator," he said. "The pointer swings on a pivot. When the end marked x points to 5, the end marked y points to 15. When x points to 8, y points to 18."

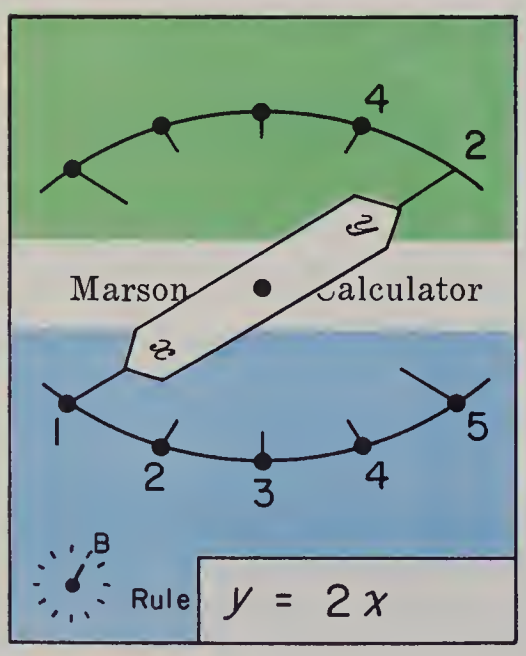
Bill volunteered, "It must be an 'add 10' machine." Do you agree?



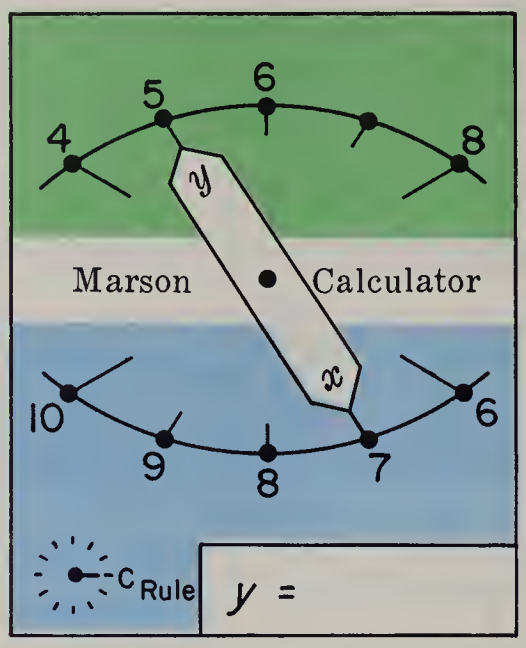
"Pointer y always points to 10 more than pointer x. In these instances, this rule applies

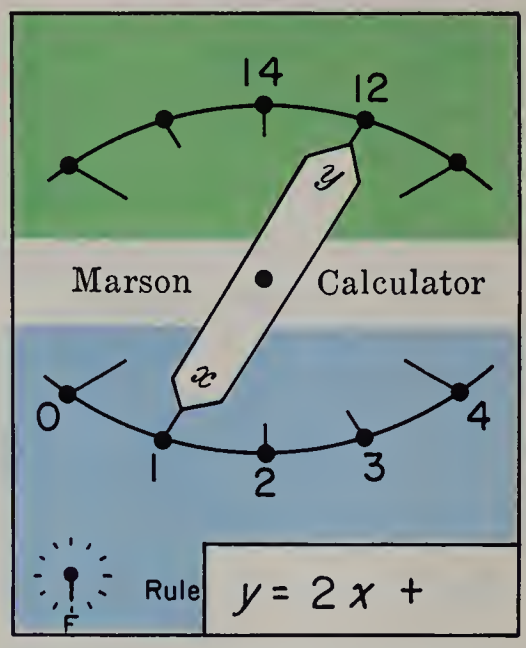
$$y = x + 10$$

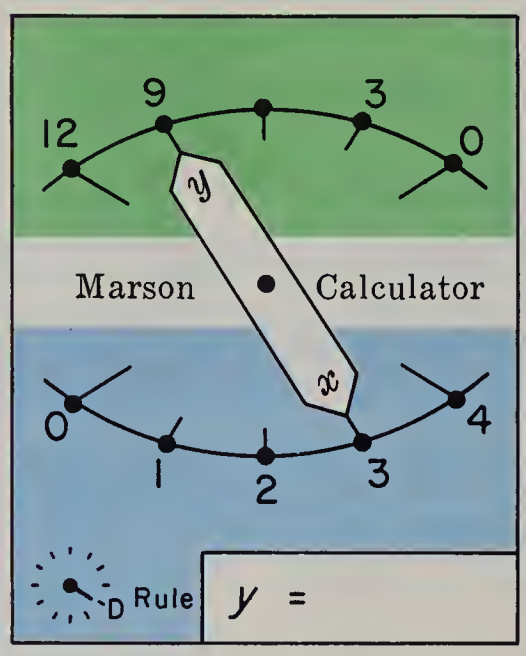
"What does the machine below calculate?" Mr. Wilson asked.

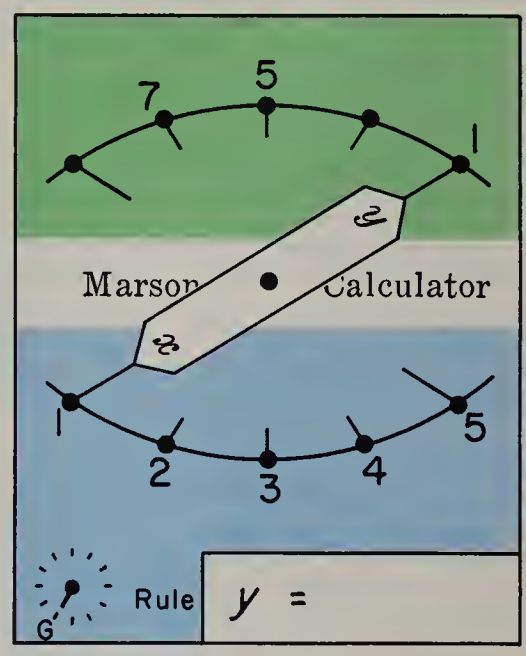


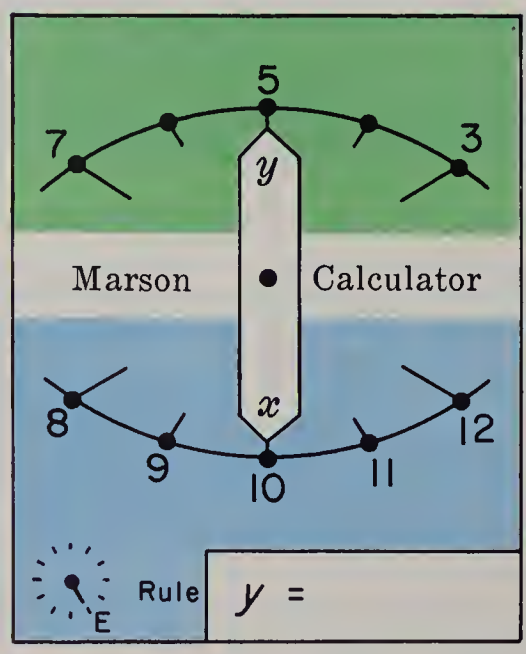
As you turn the dial in the lower left-hand corner, the rule for this calculator changes. Can you discover these rules or fill in the numbers on the calculation if anything is missing?

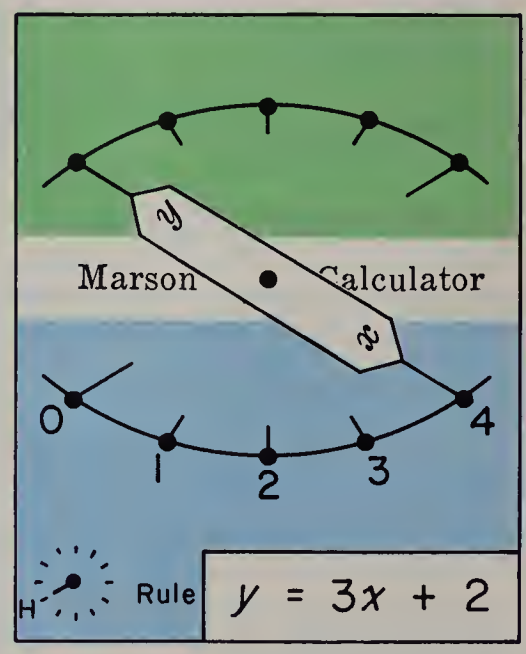












Alec has undoubtedly read Math Patterns in Science written by one of the authors, W. W. Sawyer. Alec's machine is quite similar to Professor Sawyer's machine.

The class kept a record of some of their results. They moved the pointer each time from 0 to 1 to 2 to 3, etc., in order. They noticed an interesting development no matter what rule the machine used.

They inspected the series of numbers at y. They had a common difference there.

Here are some of their records. You can complete them. (Note that C. D. is short for "common difference".)

A. Rule: $y = 2x + 1$

x	0	1	2	3	4	5	6	7
y	1	3	5	7				

C.D. 2 2 2 — — — — —

The difference between each successive y is the same — two in this case.

B. Rule: $y =$ _____

x	0	1	2	3	4	5	6	7
y	3	4	5	6				

C.D. 1 — — — — — — —

The difference between 3 and 4 is 1; the difference between 4 and 5 is 1 — and so with each succeeding pair. The common difference is 1.

C. Rule: $y =$ __ $x + 8$

x	0	1	2	3	4	5	6	7
y	8		14	17		23		

C.D. — — — — — — —

Do you see any pattern to the common difference above? Can you guess in advance what the C. D. will be in the next example?

D. Rule: $y = 5x +$ ___

x	0	1	2	3	4	5	6	7
y	10							

C.D. — — — — — — —

In some of the examples to follow, it will help if you look back to page 93 in which "directed numbers" are discussed — or "positive" and "negative" numbers.

E. Rule: $y = 6x - 1$

x	0	1	2	3	4	5	6	7
y	$^-1$							

C.D. — — — — — — —

On a number line that extends both directions from 0, how far is it from $^-1$ to $^+5$, from $^+5$ to $^+11$, etc.?

F. Rule: $y =$ _____

x	0	1	2	3	4	5	6	7
y		12		30				66

C.D — — — — — — —

G. Rule: $y = \frac{1}{2}x + 3$

x	0	1	2	3	4	5	6	7
y								

C.D — — — — — — —

Summary:

Rule A _____ → C.D. ___

Rule B _____ → C.D. ___

Rule C _____ → C.D. ___

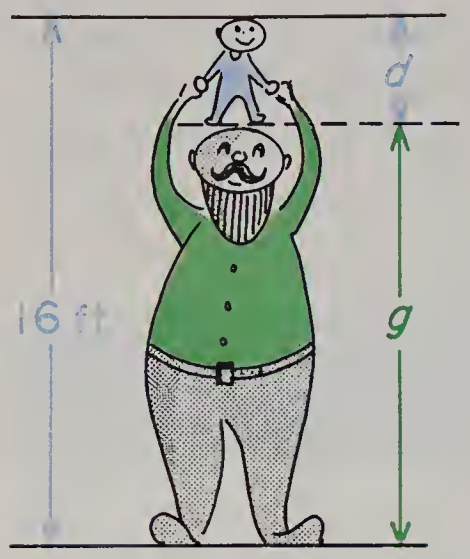
Rule D _____ → C.D. ___

Rule E _____ → C.D. ___

Rule F _____ → C.D. ___

Rule G _____ → C.D. ___

Make up examples of your own to test any pattern you notice in the summary above.



A dwarf stood on a giant's head. Together, they were 16 feet tall. What was the height of the dwarf? What was the height of the giant?

Obviously, we don't have enough information to answer the questions. However, we can fill in the chart below:

If the height of the giant and dwarf together is 16 feet

and the height of the dwarf in feet is	3	2	1	$2\frac{1}{2}$	$1\frac{1}{3}$	$\frac{7}{8}$	$\frac{11}{12}$
then the height of the giant in feet is							

In the shorthand of algebra, we can write:

If ..... $d + g = 16$
and if ..... $d = 3$
then ..... $g = 13$

or

If ..... $d + g = 16$
and ..... $d = 2$
then ..... $g = \_\_\_$

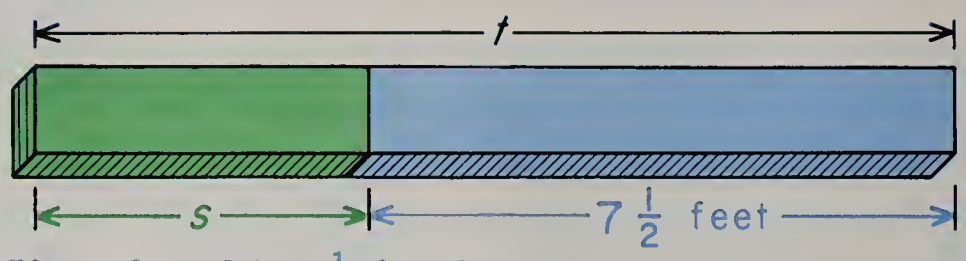
or

If ..... $d + g = 16$
and ..... $d = 1$
then ..... $g = \_\_\_$

or

$d + g = 16$
$d = 2\frac{1}{2}$
$g = \_\_\_$

Bill had a board that was $7\frac{1}{2}$ feet long. He also had a shorter board. How far would the two boards reach if placed end to end? How long is the short board?



Again, we haven't enough information but we can fill out the chart that lists some of the possibilities.

If one board is $7\frac{1}{2}$ feet long

length in feet when placed end-to-end		12		14	
length of short board in feet	$2\frac{1}{2}$		1		$3\frac{5}{8}$

In the shorthand of algebra, we can write:

If ..... $t - s = 7\frac{1}{2}$
and if ..... $s = 2\frac{1}{2}$
then ..... $t = 10$

or

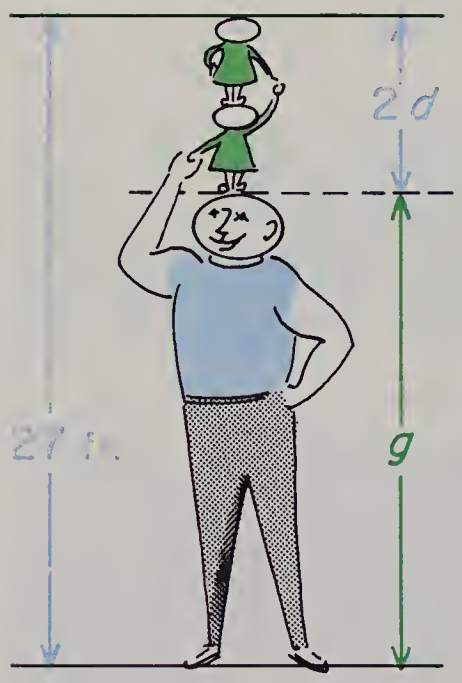
If ..... $t - s = 7\frac{1}{2}$
and ..... $t = 12$
then ..... $s = \_\_\_$

or

If ..... $t - s = 7\frac{1}{2}$
and ..... $s = 1$
then ..... $t = \_\_\_$

or

$t - s = 7\frac{1}{2}$
$s = 3\frac{5}{8}$
$t = \_\_\_$



Two dwarfs of the same size, stood, one on top of the other, on the head of a giant. The top of the head of the highest dwarf was 27 feet from the ground. Without more information, we cannot know the height of the dwarfs or the giant. But —

If a giant and 2 dwarfs are 27 feet high

and if the giant's height in feet is	20	24	$21\frac{1}{2}$	$22\frac{1}{3}$	$25\frac{1}{4}$
then each dwarf's height in feet is	$3\frac{1}{2}$				

Because:

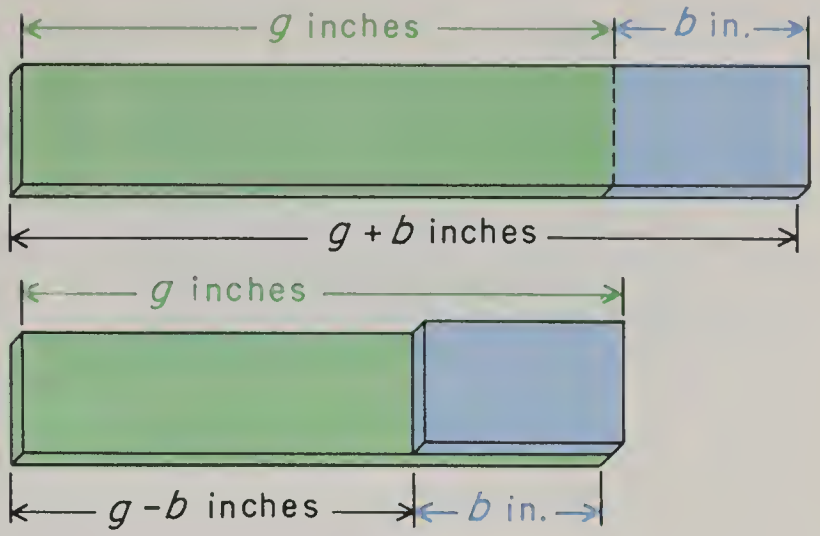
If ..... $g + 2d = 27$
and ..... $g = 20$
then ..... $2d = \_\_\_$
and ..... $d = \_\_\_$

$g + 2d = 27$
$g = 24$
$2d = \_\_\_$
$d = \_\_\_$

$g + 2d = 27$
$d = 1$
$2d = \_\_\_$
$g = \_\_\_$

Mary folded a long strip of paper and colored the region on one side of the fold green — on the other side, blue. She measured the length of the strip before it was folded — (length of green plus length of blue.) After it was folded, she measured the part of green that was not covered by the blue.

She did this with many such strips and kept a record. Can you find the length of blue and green in each example?



(All numbers indicate number of inches.)

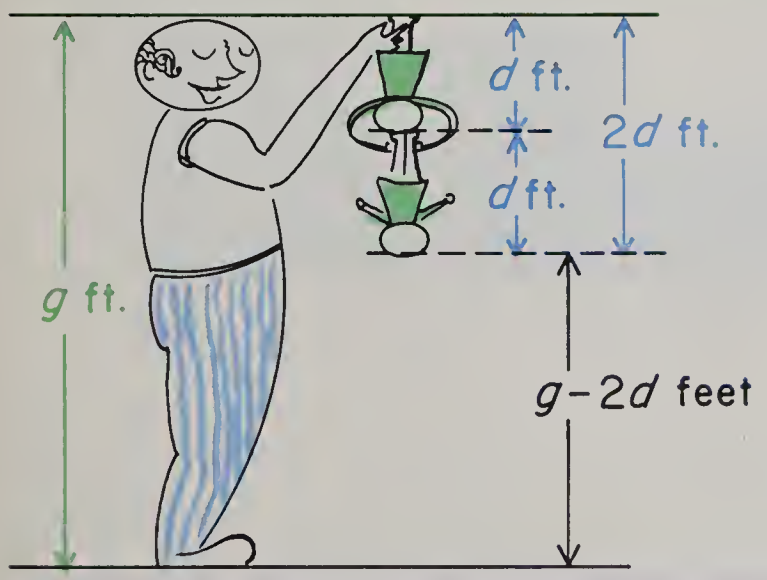
Length of green and blue together $(g+b)$	13	20	4	$15\frac{1}{2}$	$9\frac{1}{4}$			
Length of green not covered by blue $(g-b)$	7	10	2	$4\frac{1}{2}$	$4\frac{3}{4}$		4	12
Length of green (g)	10	15				18		21
Length of blue (b)	3					7	$1\frac{1}{2}$	

If .... $g + b = 13$
and .. $g - b = 7$
then ..... $g =$ ____
and ..... $b =$ ____

If .... $g + b = 20$
and ... $g - b = 10$
then ..... $b =$ ____
and..... $g =$ ____

If ..... $g = 18$
and ..... $b = 7$
then.. $g + b =$ ____
and .. $g - b =$ ____

If ..... $g = 21$
and .. $g - b = 12$
then ..... $b =$ ____
and .. $g + b =$ ____



A giant touches the ceiling with his head. He holds a dwarf's feet against the ceiling. This dwarf holds the feet of another dwarf.

The chart below indicates several possibilities. Of course, giants and dwarfs can be any height we want to imagine. All numbers are number of feet in length.

Height in feet of giant (g)	20	100	18		$33\frac{3}{4}$		$50\frac{1}{2}$	75
Height in feet of two dwarfs $(2d)$	4	3	1			9		
Height of one dwarf (d)	2	$1\frac{1}{2}$		1	$1\frac{7}{8}$			$\frac{1}{4}$
No. of feet from bottom dwarf's head to floor $(g-2d)$	16			22		38	$44\frac{1}{2}$	

If ..... $g = 100$
and ..... $2d = 3$
then ..... $d =$ ____
and.. $g - 2d =$ ____

If ..... $d = 1$
and .. $g - 2d = 22$
then ..... $2d =$ ____
and ..... $g =$ ____

If ..... $2d = 9$
and.. $g - 2d = 38$
then ..... $g =$ ____
and ..... $d =$ ____

If ..... $g = 50\frac{1}{2}$
and.. $g - 2d = 44\frac{1}{2}$
then .... $2d =$ ____
and ..... $d =$ ____

Headline

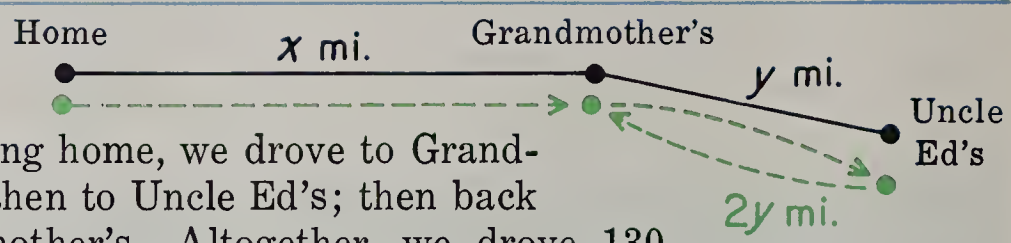
$a + b = 12$
$a - b = 2$
$a =$ ____
$b =$ ____

Example: Mrs. Clark made two purchases. Her first purchase ($\$a$) was \$2 more than the second ($\$b$). Altogether, ($a+b$ dollars), she spent \$12. The amount of her first purchase was \$_____; the second was \$_____.

Make up another example to fit the headline: _____

$x + 2y = 130$
$y =$ ____
$2y =$ ____
$x =$ ____

Example:

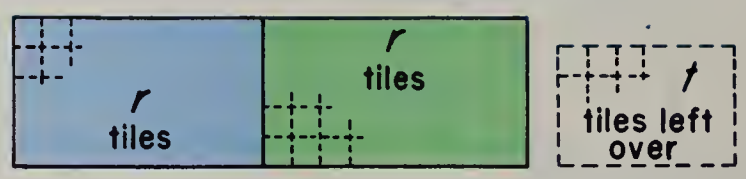


After leaving home, we drove to Grandmother's; then to Uncle Ed's; then back to Grandmother's. Altogether, we drove 130 miles. It is 20 miles from Grandmother's to Uncle Ed's. From Grandmother's, we will have to drive _____ miles to get home.

Make up another example to fit the headline: _____

$2r + t =$ ____
$t = 24$
$2r =$ ____
$r =$ ____

Mr. Brown bought 144 square tiles. He had two floors the same size to cover. When he finished, there were 2 dozen tiles left over. He used _____ tiles for each floor.



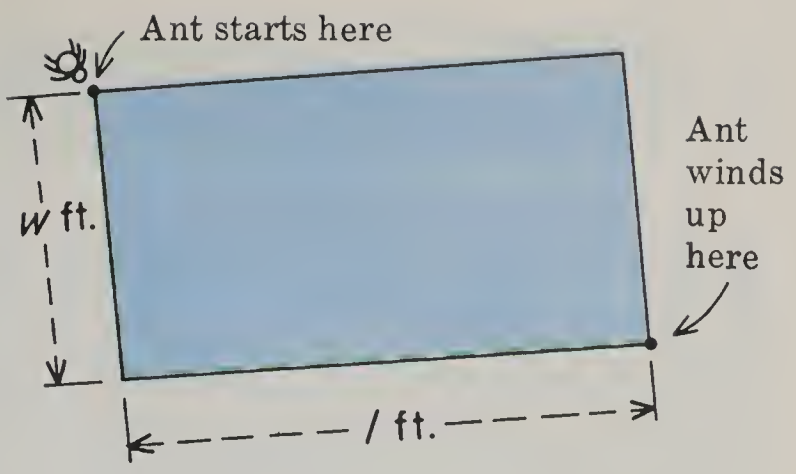
Make up another example to fit the headline: _____

$x - y = 60$
$x + 2y = 75$
$x =$ ____
$2y =$ ____
$y =$ ____

Jack and Bill each weighed the same when school started in the fall. Jack was sick for several weeks and lost some weight, and by Thanksgiving he weighed only 60 pounds. Bill gained twice as much as Jack lost, and he weighed 75 pounds. When school started, each must have weighed _____ pounds. Jack lost _____ pounds, and Bill gained _____ pounds.

Make up another example to fit the headline: _____

An ant started crawling from one corner of a rectangularly shaped room. He crawled along the wall — the width (w feet) of the room plus the length (l feet) of the room. It took exactly 360 foot square tiles to cover the floor. How far did the ant crawl?



We do not have enough information, but we can work out several of the possibilities.

Area of floor is 360 square feet, or $l \cdot w = 360$.

Length of room	30		60				72	90		
Width of room	12	15				3				
Distance ant crawled	42			38	46				49	53

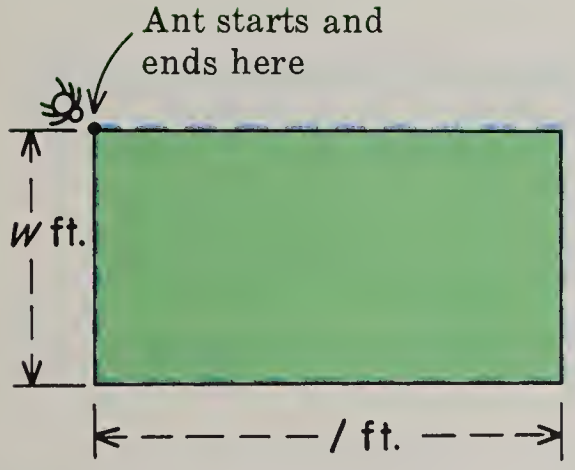
In the shorthand of algebra, we can write:

If ..... $l \cdot w = 360$
and ..... $l = 30$
then ..... $w =$ ____
and ... $l + w =$ ____

If ..... $l \cdot w = 360$
and .. $l + w = 38$
then ..... $l =$ ____
and ..... $w =$ ____

If ..... $l \cdot w = 360$
and ..... $w = 3$
then ..... $l =$ ____
and .. $l + w =$ ____

If ..... $l \cdot w = 360$
and ... $l + w = 49$
then ..... $l =$ ____
and ..... $w =$ ____



Now suppose that the ant crawls all the way around the room — back to the corner from which he started. All we know about the size of the room is that it is rectangular in shape and covered with whole tiles that are 1-foot square.

The chart below considers some of the possibilities. All numbers refer to a number of feet or square feet.

Length of room in feet (l)	10		20					
Width of room in feet (w)	15	9				8		
Area of room in square feet ($l\ w$)			340	225	77		221	143
Distance ant crawls from one corner to opposite corner ($l + w$)		21						
Twice that distance 2 ($l + w$)						34		
Total distance ant crawls ($2\ l + 2\ w$)				60			60	

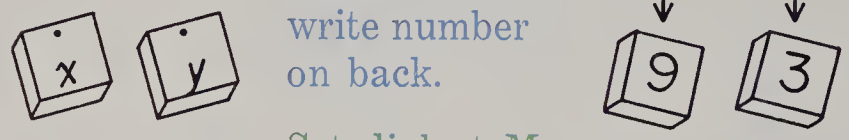
Suppose the ant crawled around three of the walls and had 11 feet to go before reaching his starting place. The trip around the three walls had been 25 feet. The room must be _____ feet wide.

In the shorthand of algebra, we could write:

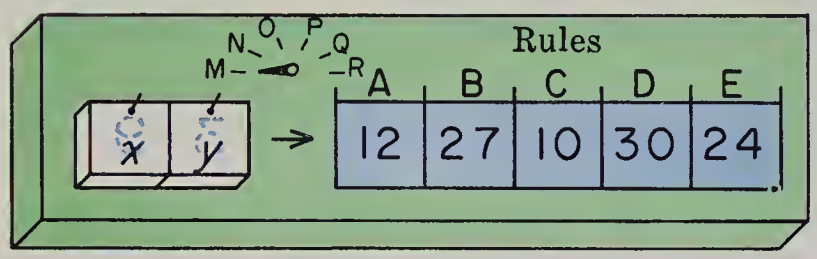
$w + l + w = 25$
$2w + l =$ ___
$l =$ ___
$2w =$ ___
$w =$ ___

Another Machine for the Science Fair left by Alec Marson.

This machine has two removable blocks marked "x" and "y." To operate it, you remove the blocks and write a number on the back; set the dial or pointer; replace the blocks — and numbers will appear immediately on each of the five blue screens.



and the machine looks like this:



Keep changing the numbers written on the blocks and keep a record of the numbers the machine writes. Perhaps you can discover the rules. Here are some results with the pointer set at M:

x , y			A	B	C	D	E
9	3	→	12	27	10	30	24
4	6	→		24		27	20
11		→		22		25	26
		→	6	5			

Complete the chart above and the rules below:

Rule A: $x, y \rightarrow x + y$

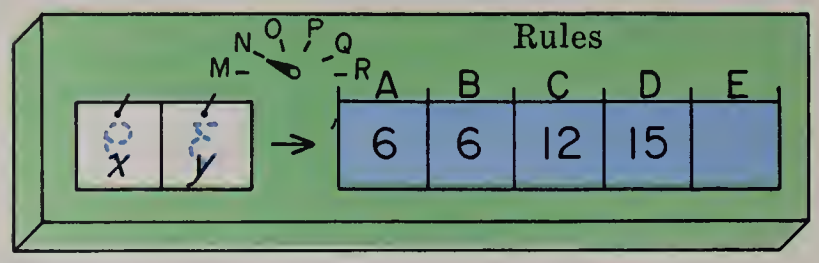
Rule B: $x, y \rightarrow$

Rule C: $x, y \rightarrow x + y - 2$

Rule D: $x, y \rightarrow$

Rule E: $x, y \rightarrow$

When the pointer was moved into position "N," the following results were noted. Tests began with "9" written on the back of "x" block and 3 on back of "y" block.



x , y			A	B	C	D	E
5	4	→	1	8		13	2
13	7	→				27	
		→	4	18			
		→		6		16	14

Rule A: $x , y \rightarrow$

Rule B: $x , y \rightarrow 2y$

Rule C: $x , y \rightarrow 2x - 2y$

Rule D: $x , y \rightarrow$

Rule E: $x , y \rightarrow 2(x - y)$

Here are some results with pointer at "0":

x , y			A	B	C	D	E
10	5	→	25	25	15	25	15
4	8	→	22		24		0
7	3	→		17	9	16	11
11		→	21	22	0	11	
	9	→	25	21		33	3

Rule A: $x , y \rightarrow x + y +$ ___

Rule B: $x , y \rightarrow 2x +$ ____

Rule C: $x , y \rightarrow$

Rule D: $x , y \rightarrow x +$ _____

Rule E: $x , y \rightarrow 2x$ _____

Here are some results with pointer at "P":

x	y		A	B	C	D	E
7	3	→	49	9	40	52	18
5	4	→		16	9	29	
8	1	→		1			
	6	→		36	64	106	
9		→			56	86	50

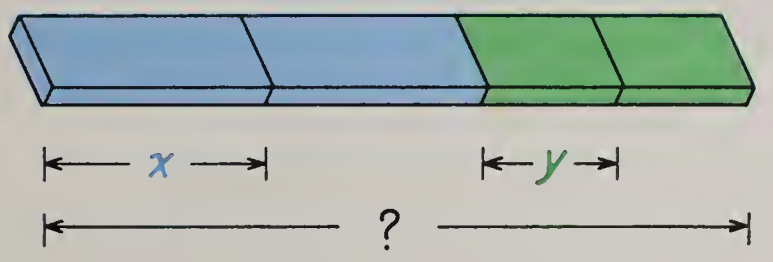
Rule A $x, y \rightarrow x \cdot x$ or x^2

Rule B $x, y \rightarrow$ ____

Rule C $x, y \rightarrow x^2 -$ ____

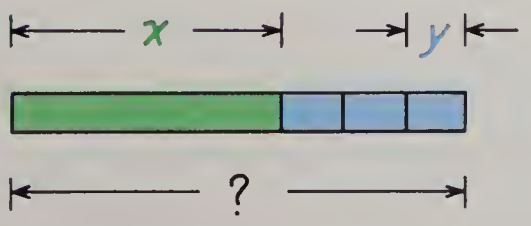
Rule D $x, y \rightarrow x^2 +$ ____

Rule E $x, y \rightarrow 2 \cdot y^2$ or $2y^2$



This suggests the idea of _____

This sketch suggests the idea of _____



The sketch above suggests the idea of _____

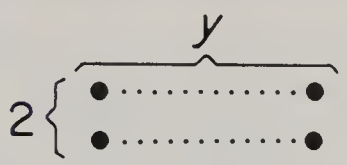
In the blue-tinted area are some expressions we have been using to complete the rules on these pages.

x^2 $\quad x + 3y \quad$ $x^2 - y^2$

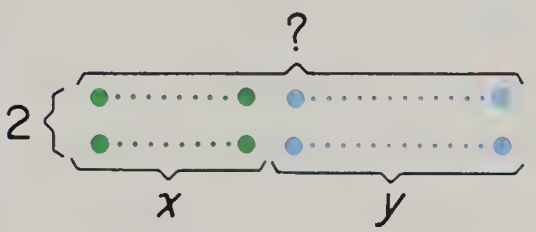
$2x + 2y \quad y^2 \quad x + y - 2$

$2y \quad 2(x + y)$

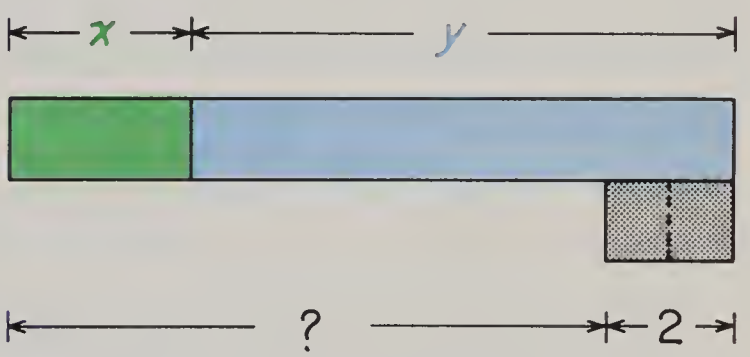
Find sketches below that suggest these ideas



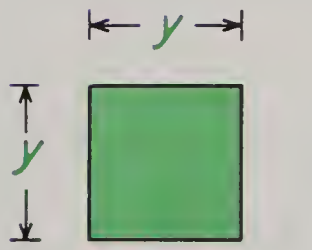
This sketch suggests the idea of _____



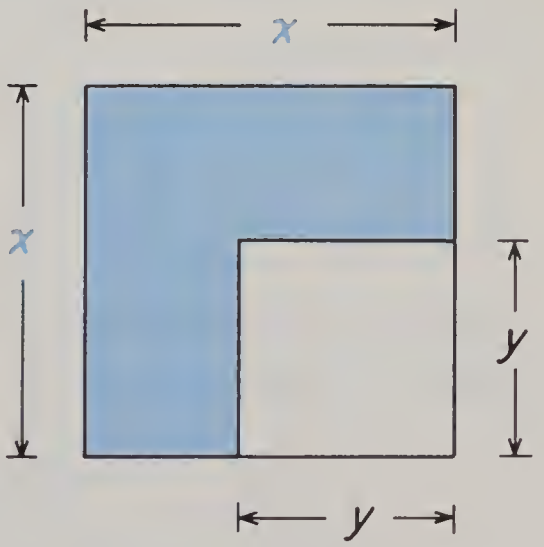
This suggests the idea of _____



The above sketch suggests the idea of _____



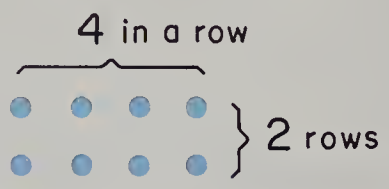
The sketch above suggests the idea _____



The blue part of the sketch above suggests the idea of _____

Sketches can illustrate many ideas suggested by the shorthand of algebra.

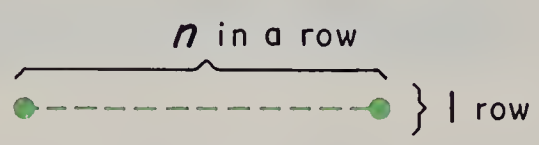
In arithmetic, we use sketches or diagrams to see clearly what is going on. For example: there are four trees in each of two rows. A sketch might be:



Because $2 \times 4 = 8$, there would be eight trees in all.

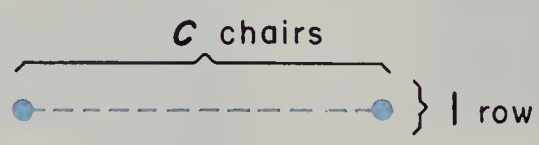
In some situations we don't have enough information to draw such a complete sketch. However, we can indicate at least as much as we do know. Here is an example:

Think of a number. Arrange that many counters in a single row. Only you know what number you thought of. Others can use a letter such as "***n***" to take the place of that number until they find out for sure what it is. Here is a sketch of the idea:

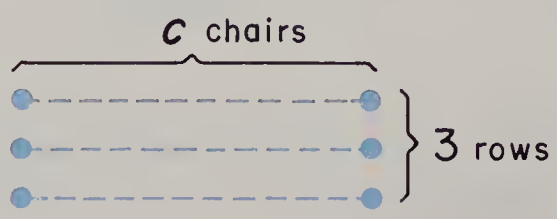


This sketch will serve well unless the number you thought of was 1 or 0.

Think once again of a number. Arrange that many chairs in a row. We can say that you have arranged ***c*** chairs in a row, and sketch it as:

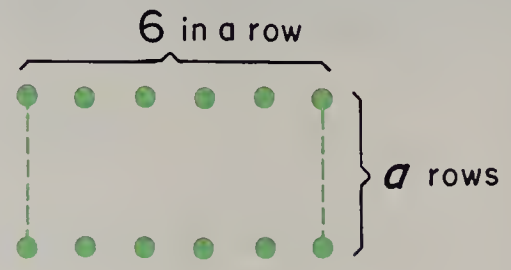


Next, arrange that same number of chairs in each of three rows.



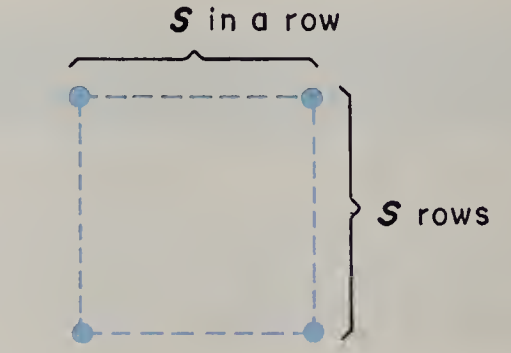
We can say that you arrange "***c***" chairs in each of 3 rows. So, there are $\mathbf{3} \times \boldsymbol{c}$ chairs or $\mathbf{3} \cdot \boldsymbol{c}$ or $\mathbf{3}\,\boldsymbol{c}$ chairs.

Think of another number. Draw that number of rows with 6 dots in each row. "***a***" will remind us of the number you thought of.



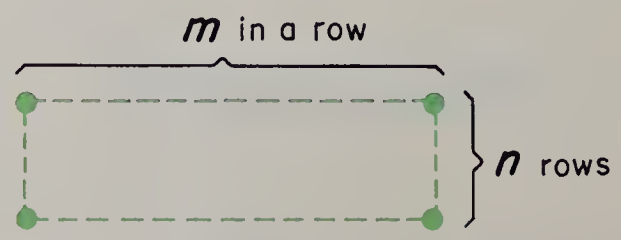
You must have drawn $\mathbf{6} \times \boldsymbol{a}$ or $\mathbf{6}\,\boldsymbol{a}$ dots.

Think of a number. (We will use ***s*** to remind us of that number until we know more about it.) Put that number of counters in a row. Add rows until you have that same number of rows.



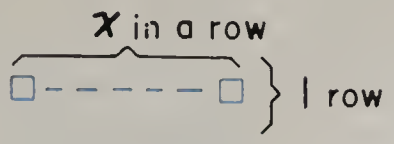
You must have, in all, $\boldsymbol{s} \times \boldsymbol{s}$ counters or $\boldsymbol{s}^2$ counters. "$\boldsymbol{s}^2$" is said as "s squared."

Think of a number. (We'll think of it as ***m***.) And, once again, think of a number. (We'll use ***n*** to remind us of it.) Now draw ***n*** rows of dots with ***m*** dots in each row.

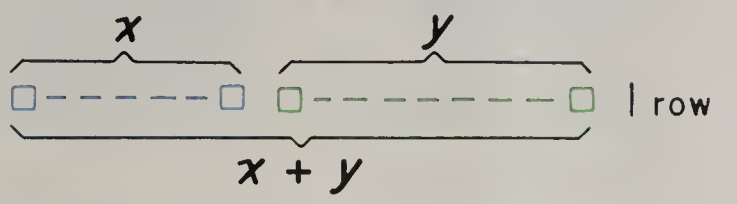


This sketch reminds us that you must have drawn $\boldsymbol{m} \times \boldsymbol{n}$ or $\boldsymbol{mn}$ dots.

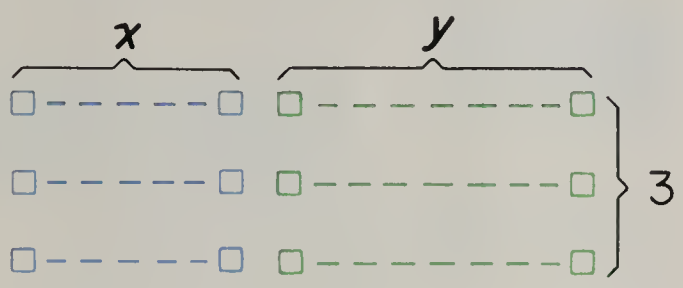
Think of a number. Draw that many blue squares in a row.



Think of another number. Extend the row with that number of green squares.

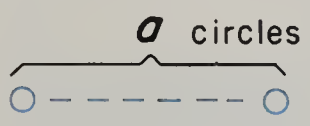


There are $x + y$ squares in a row. Now, draw 3 such rows.

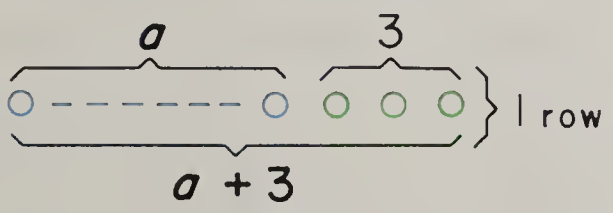


The sketch tells us you must have drawn $3x + 3y$ squares. Or, we can say, there must be $3\ (x + y)$ squares.

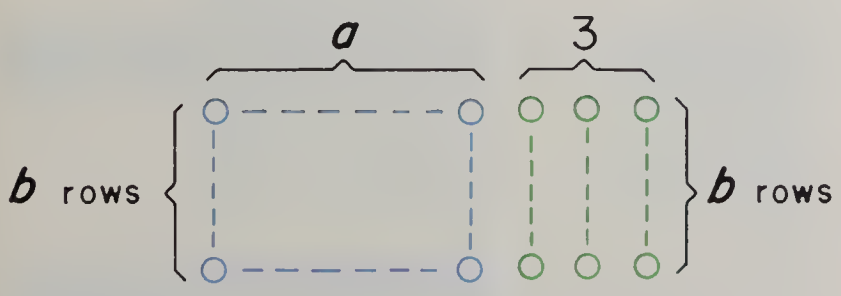
Think of a number and draw a row with that number of circles in it.



add 3 circles to that row



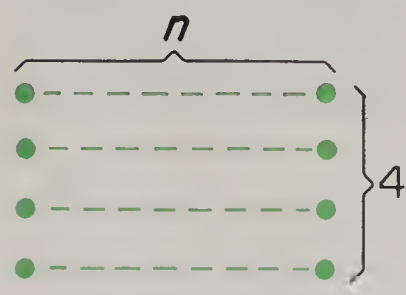
You must have drawn $a + 3$ circles. Now think of a number (b) and draw that number of rows like the one above.



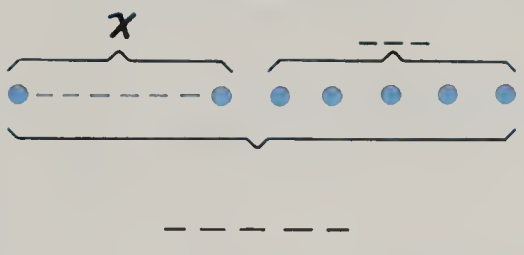
There are $a \times b$ or ab blue circles. There are $3 \times b$ or $3b$ rows. There are $ab + 3b$ altogether.

Or, we can say that altogether there are $b\ (a + 3)$ circles.

Each diagram below helps indicate a certain idea. Complete the sketch and the shorthand that suggests the same idea.

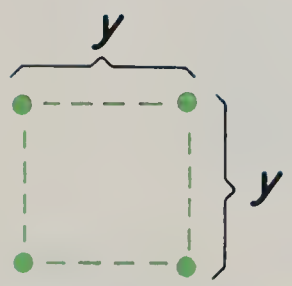


4 x n or _____

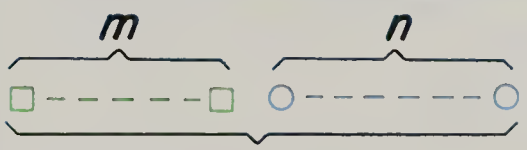


x + _____

_____

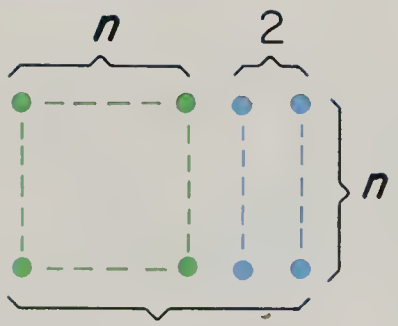


y · y or _____



_____

_____

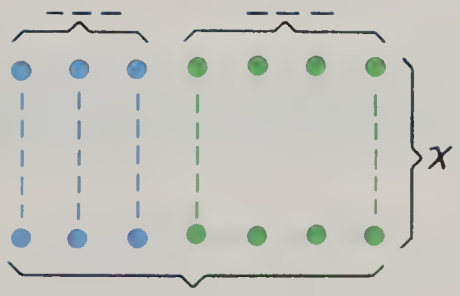


n^2 + _____

or

n (_____)

_____



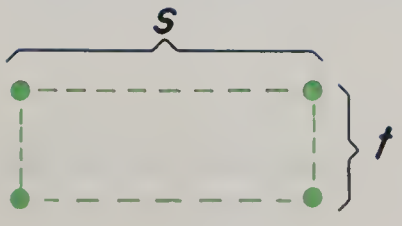
3 x + _____

or

x (+)

or 7 x

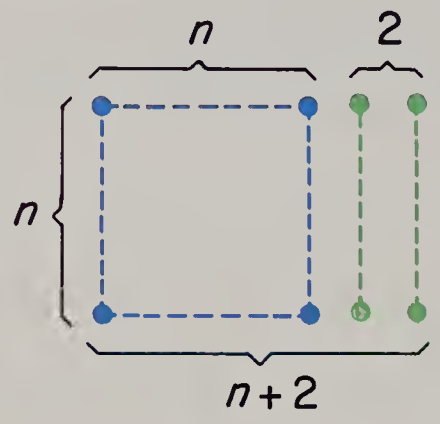
_____



_____

Jane said to Al, Jack, Betty, Dick and Ruth, "Each of you think of a number. Draw a row with that number of dots. Now add two more dots to the row.

"Think again of the number you first thought of and draw that many rows just like your first row. I don't know what you've drawn, but a sketch helps me see clearly what I do know.



"I use n to remind me of whatever number you thought of. So, I can say that you have drawn $n^2 + 2n$ dots. Or, I can say you have drawn $n\ (n + 2)$ dots.

"Each of you fill in this chart, and see if I am not right."

	Al	Jack	Betty	Dick	Ruth
Your number n	5	8			20
n^2	25			49	
$2n$	10		22		
$n^2 + 2n$					
n	5				
$n + 2$					
$n(n + 2)$					

"No matter what number you think of," Jane explained, "my sketch remains the same. And it tells me that —

$$n\,(n + 2) = n^2 + 2n$$

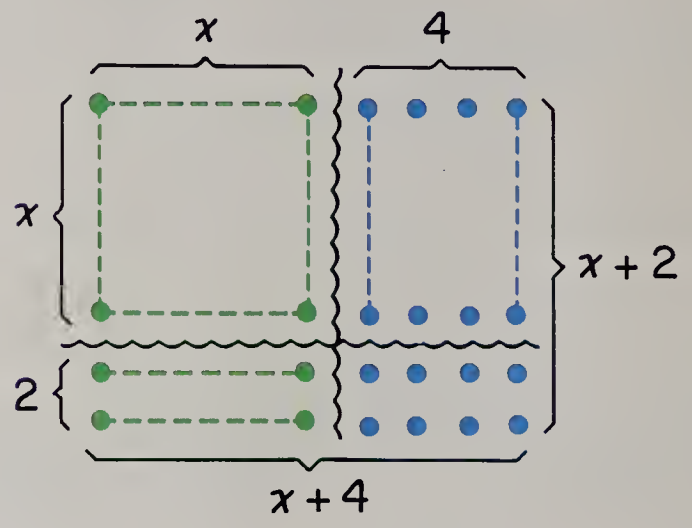
"Try it with any number you can think of. The results will be the same."

Jane asked her friends to try another experiment with her.

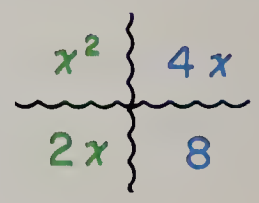
"Think of another number. I'm going to use x to remind me of it. Draw that many dots in the top row. Then add 4 more.

"Next, draw as many rows like your first row as the number you first thought of.

"My sketch looks much like an earlier one. But, now add two more rows. There is my sketch:



"Another diagram leads me to the conclusion that —



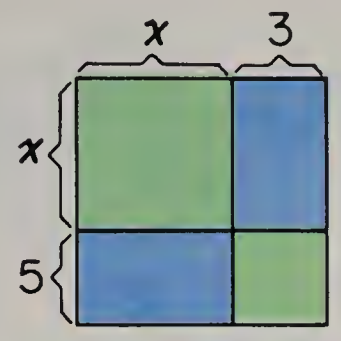
$$(x + 2)(x + 4) = x^2 + 6x + 8$$

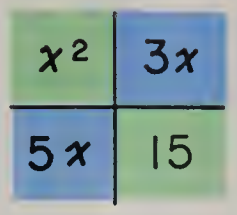
"Test my conclusion in the chart below."

	Al	Jack	Betty	Dick
x	6			
$x + 2$		11		
$x + 4$				
$(x + 2)(x + 4)$			35	
x^2				25
$6x$				
$x^2 + 6x + 8$				

Make your own chart like the one above. Think of other numbers. Keep experimenting until you are completely confident that there will be no surprises.

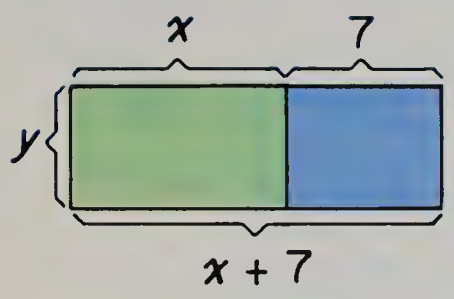
Drawing dots gets tiresome. Another kind of sketch might look like this:

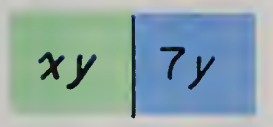




This suggests that —

$(x + 3)\ (x + 5) = x^2 + \_\_\_\ x + \_\_\_\_$

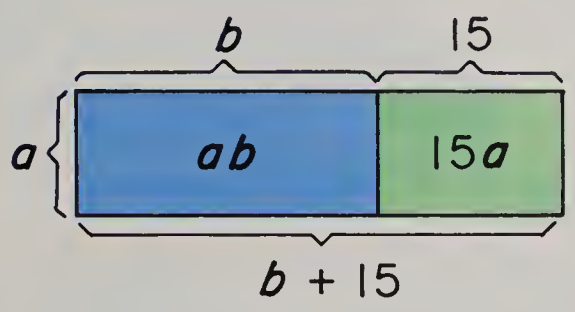




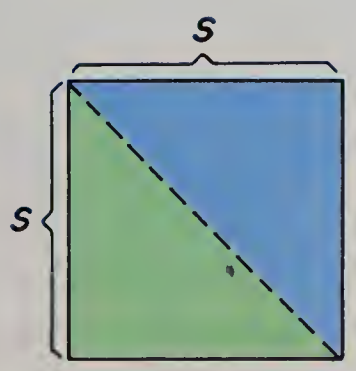
This suggests that —

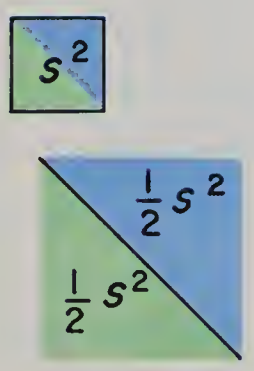
$xy + \_\_\_\_ = y\ (\_\_\_ + \_\_\_)$

We can simplify the sketch still further:



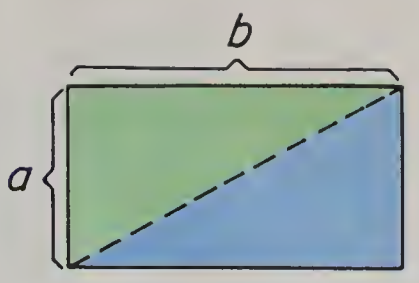
$ab + \_\_\_ = a\ (\_\_\_ + \_\_\_)$

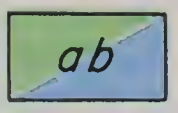


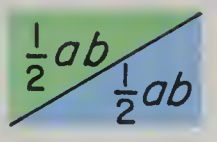


$\frac{1}{2}\ s^2 + \frac{1}{2}\ \_\_\_ = \_\_\_$

The blue and green regions together suggest the idea of s^2. The green region suggests the idea of $\frac{1}{2}s^2$. The blue region suggests the same idea.

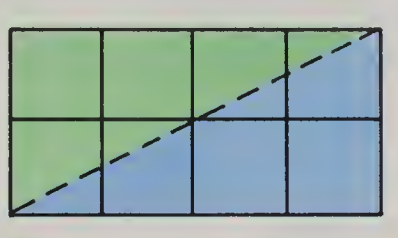




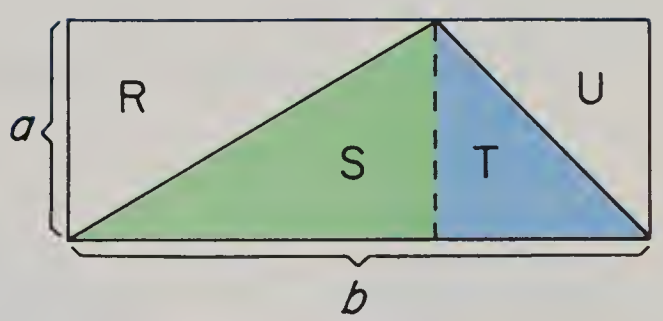


$\frac{1}{2}\ \_\_\_ + \frac{1}{2}\ \_\_\_ = \_\_\_\_$

a and ***b*** remind us of two unknown numbers. If we had known that those numbers were 2 and 4, we could have drawn the following sketch:



It would take $\frac{1}{2}$ of 2×4 small squares to cover the green region — 4 squares; and $\frac{1}{2}$ of 2×4 to cover the blue region. This is a special example of the idea of the sketch at the top of this column.



The large rectangle is a sketch that suggests the idea of ***ab***.

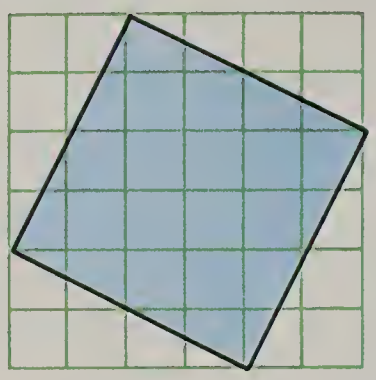
Is the region labeled R the same size as the green region labeled S? ____

Is the region labeled U the same size as the blue region labeled T? ____

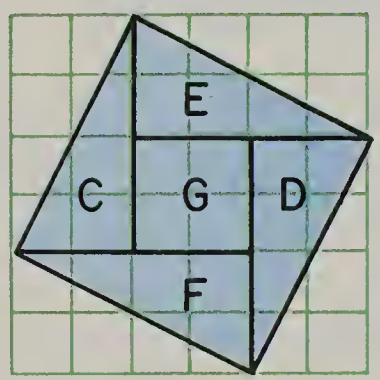
Are regions R and U together the same size as regions S and T together? ____

Can we say that the colored region (blue and green together) suggests the idea of $\frac{1}{2}\ ab$? ____

How many small squares ☐ would you need to cover the large blue square below? (You would need to cut some up in odd shapes.)

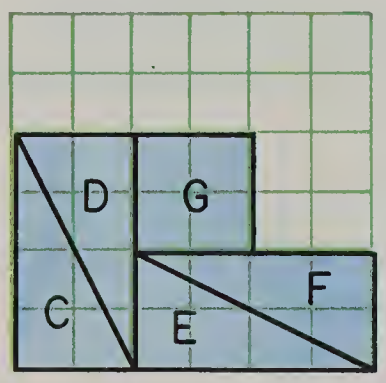


One way to think about the question is to see the square as made up of four triangles and a small square.



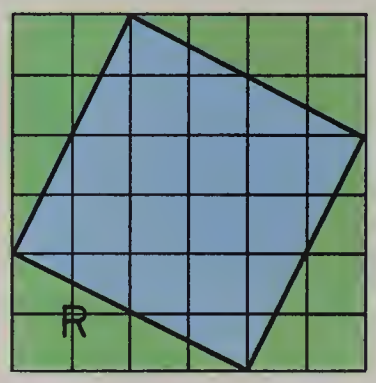
Each blue region is labeled so we can talk about it.

These four triangles and the square can be rearranged as in the following diagram:

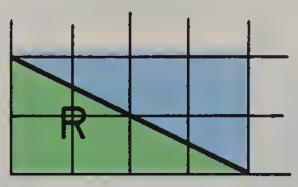


Now you can easily count the number of small squares you would need to cover the blue region.

Here is another approach to the same problem. In the entire diagram, there are 6×6 small squares, or 36. If we can find the number of squares we would need to cover the green regions below, we can get to our result. We would simply subtract the number of whole squares that are green from 36. That will tell us all we need to know about the blue squares.



The four green regions are the same size. So, let's consider only one.



The blue and green shaded areas exactly cover the eight squares. So it must take four square pieces to cover R.

$4 \times 4 = 16$

$36 - 16 = 20$

So again we have found the answer to our question. We would be surprised if our results in both cases had been unlike each other.

On the next page you can use either method you like to find the size of the large blue areas or regions in each example.

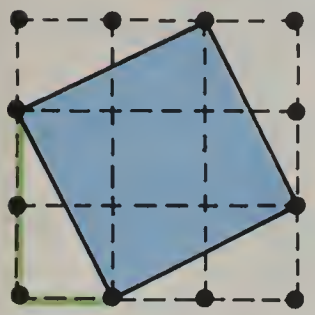
After you have completed most of the chart at the bottom of the next page, look for an important pattern.

In each column, how can you know the result that will appear in the bottom row from the 2 results above it?

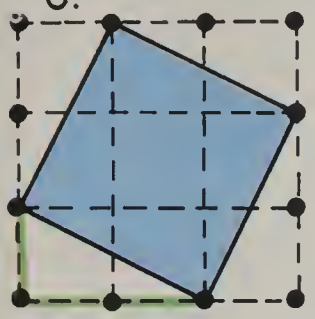
Complete record below. Indicate length of green lines in units of •—•. Record number of small squares required to cover each large blue square.

a.

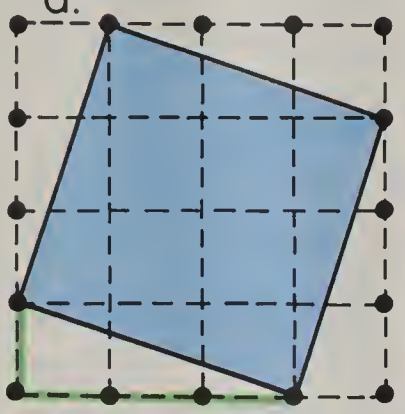
b.



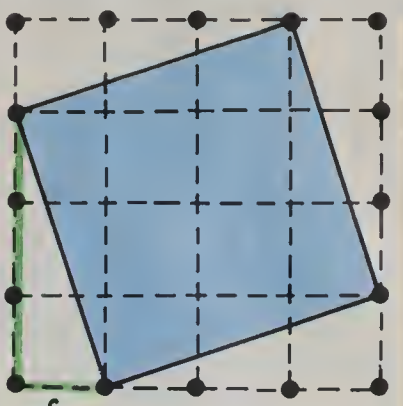
c.



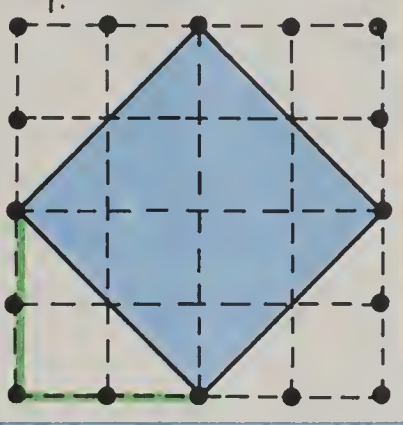
d.



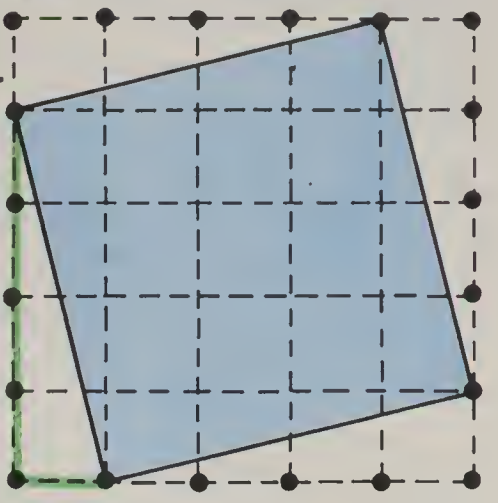
e.



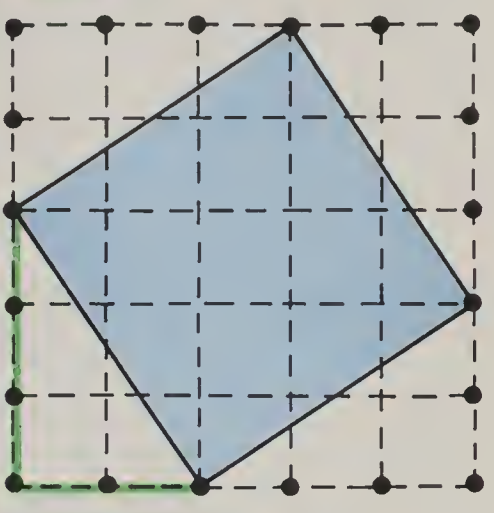
f.



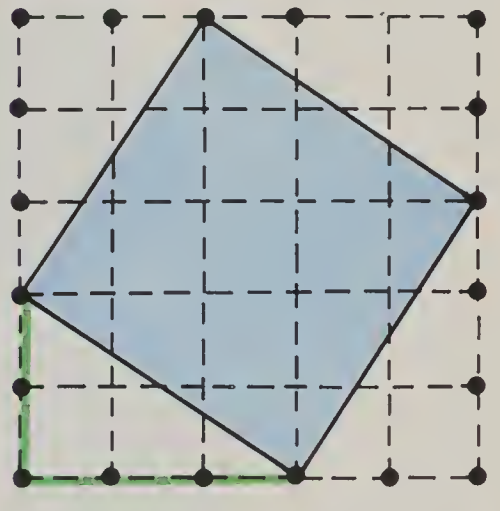
g.



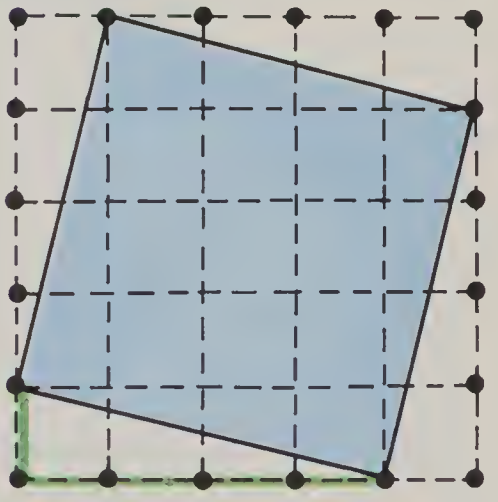
h.



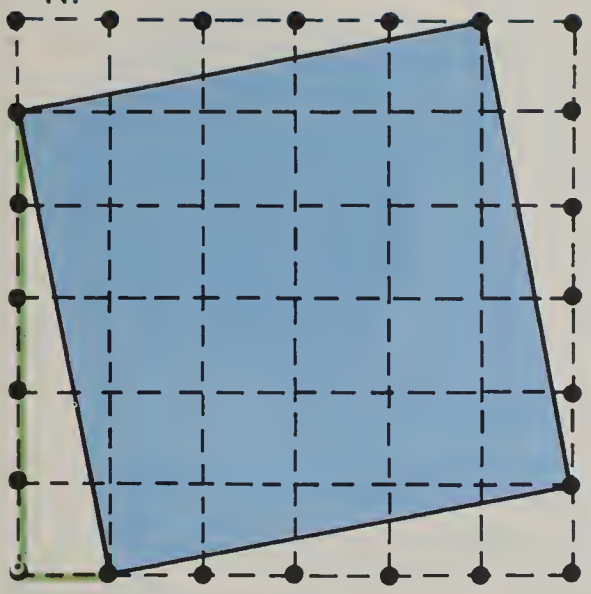
i.



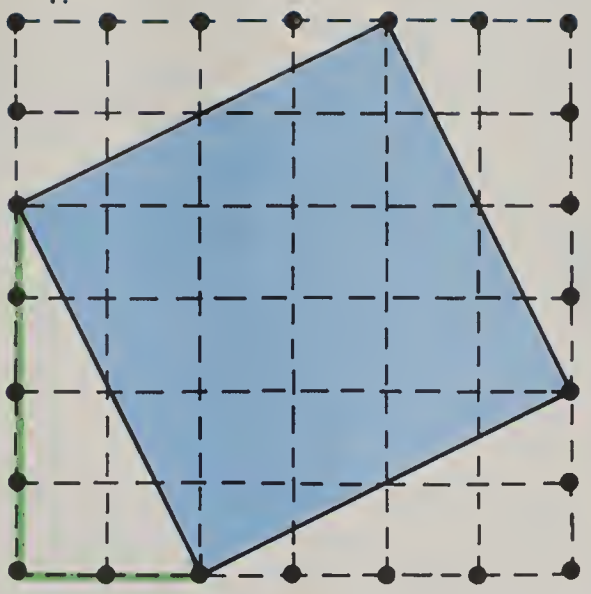
j.



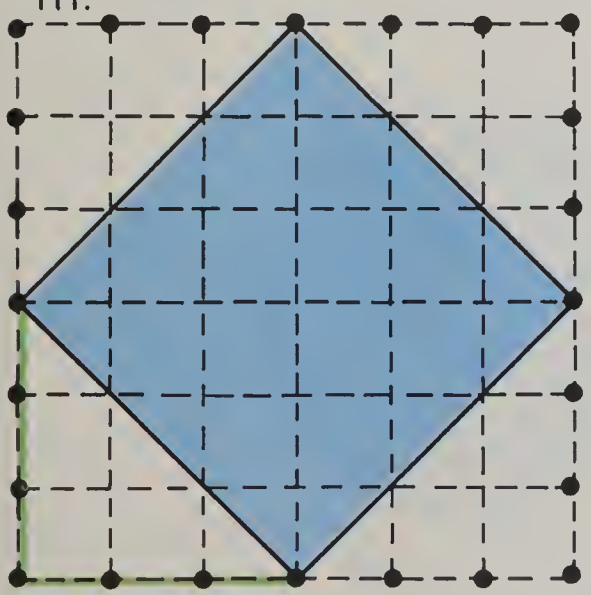
k.



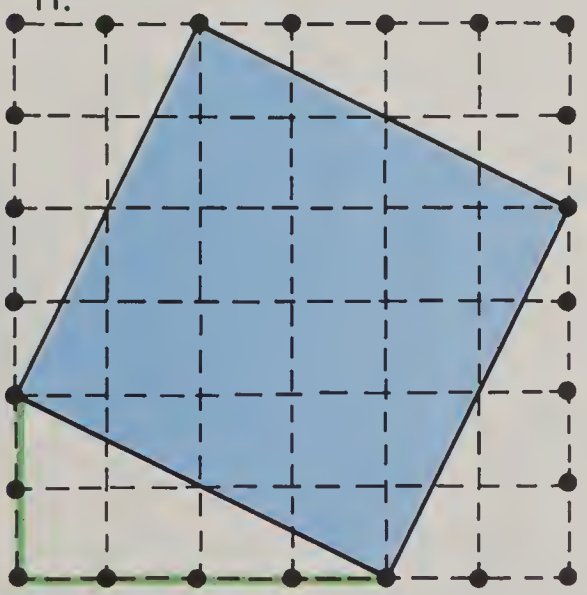
l.



m.

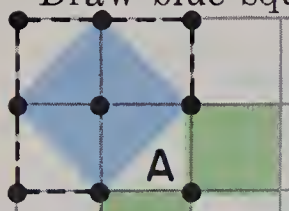


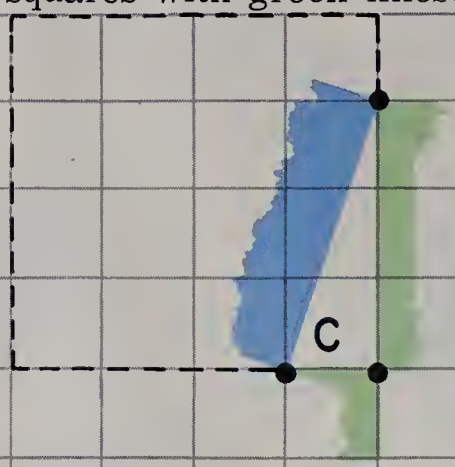
n.



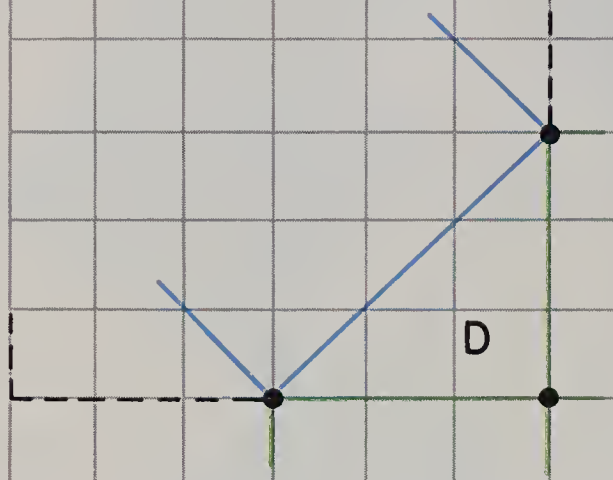
	a	b	c	d	e	f	g	h	i	j	k	l	m	n	x	y	z
No. of •—•	1	2													6	7	3
No. of •—•	1	1													8	2	4
Squares shaded blue	2	5															

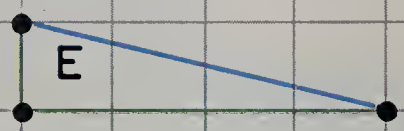
Draw blue squares with blue lines as sides. Draw green squares with green lines.

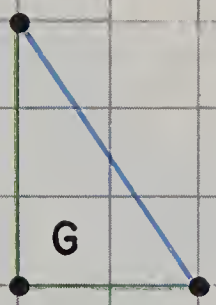


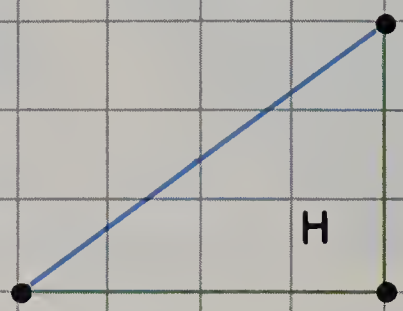


With a broken line, draw a square that is touched by all four corners of each blue square as in examples A, B, and C above.









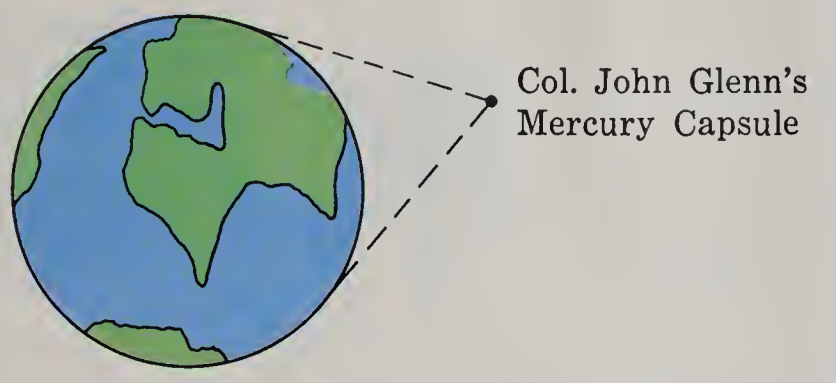
	A	B	C	D	E	F	G	H	V	W
No. top or bottom	1	4							36	
No. on side	1									25
No. in blue □	2	8							52	50

Before we start, you must realize that we will, of course, run into some very large distances — and some very large numbers.

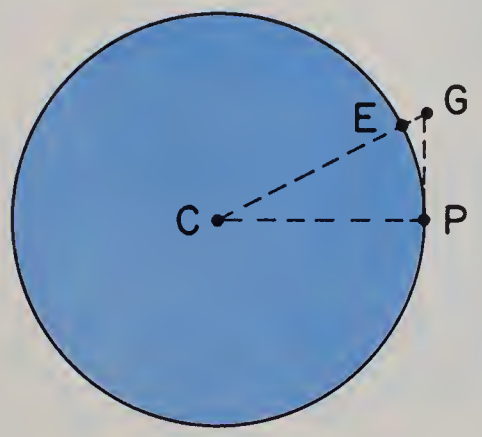
Col. John Glenn was the first American astronaut to orbit the world. Some of the time, he was 100 miles above the earth. How far do you think he could see in all directions?

Let's see if we can find out!

Here is a sketch that will help us find out (our sketch is not to scale).

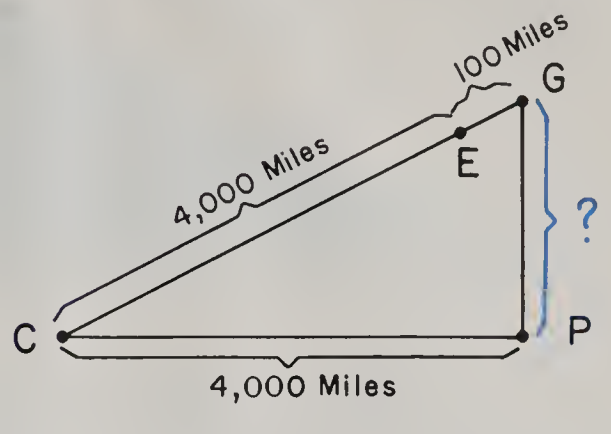


If we took a slice right through the center of the world, we would see the problem this way:



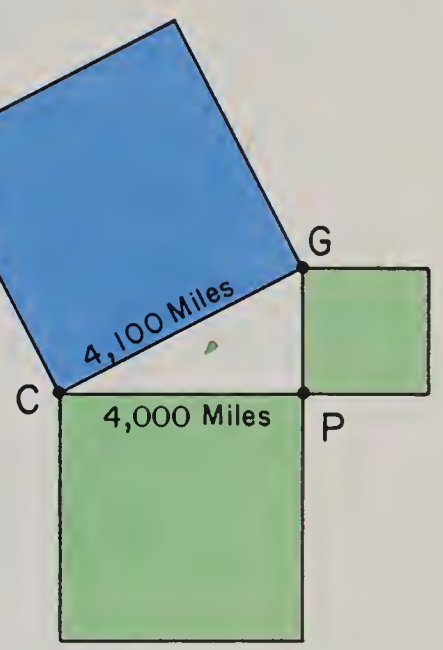
G suggests Glenn, and he is 100 miles above the earth at point E. From that height, he can see as far as point P on the surface of the earth. Of course, point E and point P are about the same distance from C — the center of the earth — and it is about 4,000 miles from any point on the earth to the center of the earth.

Let's draw a larger sketch of that part of our diagram we are most interested in and fill in some distances.



How far is Glenn from the center of the earth? (4,000 + 100 = _____.)

He can see from G to P. How far is that?



If the blue square is 4,100 miles on each side, how many square miles are there in the blue square? _____

How many square miles in the large green square? _____

How many square miles in the small green square? _____

How long is each side of the small green square? _____

That length is the distance Col. Glenn can see. (Find out yourself before turning the page.)

Marson
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EARTH

Dear Mrs. Esser,

I'm sorry I had to leave without saying goodbye. On my way home, I worked out the answer to that problem about how far Col. Glenn could see from 100 miles out in space. Here is my computation:

$$\begin{array}{r} 4{,}100 \\ \times\ 4{,}100 \\ \hline 410000 \\ 16400 \\ \hline 16{,}810{,}000 \end{array}$$

$$\begin{array}{r} 4{,}000 \\ \times\ 4{,}000 \\ \hline 16{,}000{,}000 \end{array}$$

$$\begin{array}{r} 16{,}810{,}000 \\ 16{,}000{,}000 \\ \hline 810{,}000 \end{array}$$

$810{,}000 = 900 \times 900$

So, Col. John Glenn could see about 900 miles on Earth in every direction.

This means he could look over about 2,500,000 square miles of Earth scenery.

Your friend,

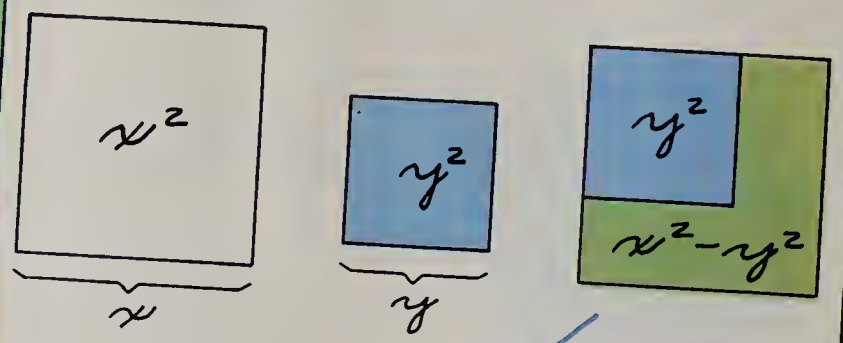
Alec Marson

...se, I could have
...rance Col. Glenn
... much easier
...learned in Algebra.
I learned that, for any numbers (I use x and y here as reminders) it is true that—

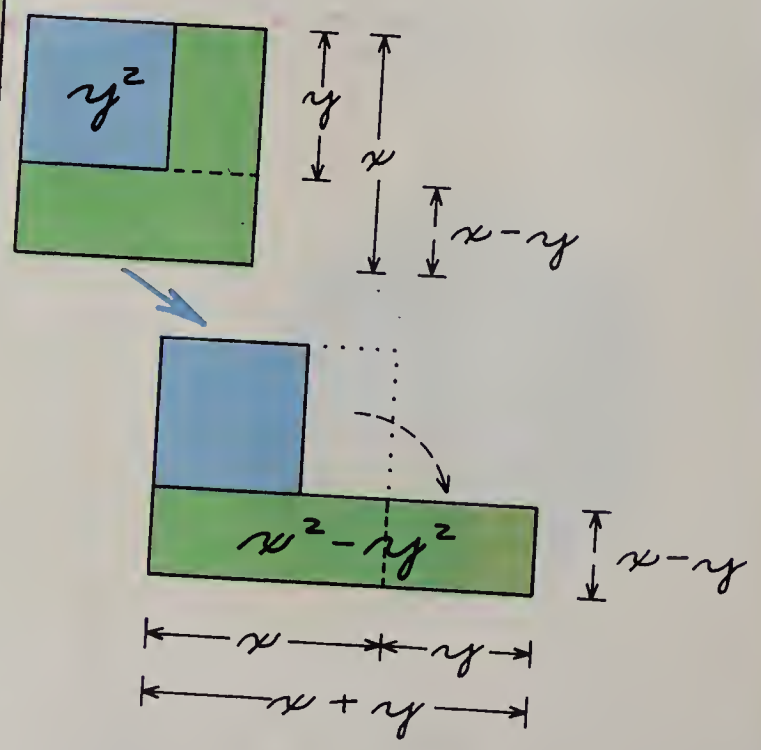
$$x^2 - y^2 = (x+y)\cdot(x-y)$$

$$4{,}100^2 - 4{,}000^2 = (8{,}100)\cdot(100)$$
$$= 810{,}000$$
$$= 900 \times 900$$

Here is a series of sketches I use to help me remember and understand that amazing shortcut.



The green region suggests the idea of x^2-y^2. How big is it?



The green region certainly suggests the idea of $(x+y)\cdot(x-y)$. Clearly $x^2-y^2=(x+y)(x-y)$.

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